

# GEOMETRY OF GENERATING FUNCTIONS AND LAGRANGIAN SPECTRAL INVARIANTS

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**ABSTRACT.** Partially motivated by the study of topological Hamiltonian dynamics, we prove various  $C^0$ -aspects of the Lagrangian spectral invariants and the basic phase functions  $f_H$ , that is, a natural graph selector constructed by Lagrangian Floer homology of  $H$  (relative to the zero section  $o_N$ ). In particular, we prove the inequality  $\rho^{lag}(H; 1) \geq \max f_H$  and a  $C^0$ -continuity result of the Lagrangian spectral capacity

$$\gamma(\phi_H^1(o_N)) := \rho^{lag}(H; 1) - \rho^{lag}(H; [pt]^\#)$$

which reads that  $\gamma(\phi_H^1(o_N)) \rightarrow 0$  as  $\text{osc}_{C^0}(\phi_H^1; o_N) \rightarrow 0$ . We also prove that the micro-support of the singular locus  $\text{Sing}(\sigma_H)$  of  $f_H$  is ruled by affine lines conormal to the top stratum of  $\text{Sing}(\sigma_H)$ . Then using this structure theorem, we define the notion of *cliff-wall surgery* (for generic  $H$ 's) which replaces multi-valued Lagrangian graph  $\phi_H^1(o_N)$  by a rectifiable Lagrangian cycle that is canonically constructed out of the single valued branch  $\Sigma_H := \text{Graph } df_H \subset \phi_H^1(o_N)$  associated to the graph selector  $f_H$ .

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## 1. INTRODUCTION

We always assume that the ambient manifolds  $M$  or  $N$  are connected throughout the entire paper.

**1.1. Hamiltonian topology.** In [OM], Müller and the author introduced the notion of Hamiltonian topology on the space

$$\mathcal{P}^{ham}(Symp(M, \omega), id)$$

of Hamiltonian paths  $\lambda : [0, 1] \rightarrow Symp(M, \omega)$  with  $\lambda(t) = \phi_H^t$  for some time-dependent Hamiltonian  $H$ . We would like to emphasize that we do *not* assume that  $H$  is normalized *unless otherwise said explicitly*. This is because we need to consider both compactly supported and mean-normalized Hamiltonians and suitably transform one to the other in the course of the proof of the main theorems of this paper. One novelty of the present paper is an extensive and careful usage of the normalization constants of the Hamiltonian which naturally arise in various contexts in the course of the proof of the main theorems. It turns out that this analysis of the normalization constants is one of the crucial elements in the proofs of various results in the present paper.

In this subsection, we first recall the definition of this Hamiltonian topology.

We start with the case of closed  $(M, \omega)$ . For a given continuous function  $h : M \rightarrow \mathbb{R}$ , we denote

$$\text{osc}(h) = \max h - \min h.$$

We define the  $C^0$ -distance  $\bar{d}$  on  $Homeo(M)$  by the symmetrized  $C^0$ -distance

$$\bar{d}(\phi, \psi) = d_{C^0}(\phi, \psi) + d_{C^0}(\phi^{-1}, \psi^{-1})$$

and the  $C^0$ -distance, again denoted by  $\bar{d}$ , on

$$\mathcal{P}^{ham}(Symp(M, \omega), id) \subset \mathcal{P}(Homeo(M), id)$$

by

$$\bar{d}(\lambda, \mu) = \max_{t \in [0, 1]} \bar{d}(\lambda(t), \mu(t)).$$

The Hofer length of Hamiltonian path  $\lambda = \phi_H$  is defined by

$$\text{leng}(\lambda) = \int_0^1 \text{osc}(H_t) dt = \|H\|.$$

Following the notations of [OM], we denote by  $\phi_H$  the Hamiltonian path

$$\phi_H : t \mapsto \phi_H^t; [0, 1] \rightarrow \text{Ham}(M, \omega)$$

and by  $\text{Dev}(\lambda)$  the associated normalized Hamiltonian

$$\text{Dev}(\lambda) := \underline{H}, \quad \lambda = \phi_H \quad (1.1)$$

where  $\underline{H}$  is defined by

$$\underline{H}(t, x) = H(t, x) - \frac{1}{\text{vol}_\omega(M)} \int_M H(t, x) \omega^n. \quad (1.2)$$

We normalize  $\omega$  so that  $\text{vol}_\omega(M) = \int_M \omega^n = 1$  but do not remove the normalizing factor  $\frac{1}{\text{vol}_\omega(M)}$  to make the meaning of  $\underline{H}$  more conspicuous.

**Definition 1.1.** Let  $(M, \omega)$  be a closed symplectic manifold. Let  $\lambda, \mu$  be smooth Hamiltonian paths. The *Hamiltonian topology* is the metric topology induced by the metric

$$d_{\text{ham}}(\lambda, \mu) := \bar{d}(\lambda, \mu) + \text{leng}(\lambda^{-1}\mu). \quad (1.3)$$

Now we recall the notion of topological Hamiltonian flows and Hamiltonian homeomorphisms introduced in [OM].

**Definition 1.2** ( $L^{(1,\infty)}$  topological Hamiltonian flow). A continuous map  $\lambda : \mathbb{R} \rightarrow \text{Homeo}(M)$  is called a topological Hamiltonian flow if there exists a sequence of smooth Hamiltonians  $H_i : \mathbb{R} \times M \rightarrow \mathbb{R}$  satisfying the following:

- (1)  $\phi_{H_i} \rightarrow \lambda$  locally uniformly on  $\mathbb{R} \times M$ .
- (2) the sequence  $H_i$  is Cauchy in the  $L^{(1,\infty)}$ -topology locally in time and so has a limit  $H_\infty$  lying in  $L^{(1,\infty)}$  on any compact interval  $[a, b]$ .

We call any such  $\phi_{H_i}$  or  $H_i$  an *approximating sequence* of  $\lambda$ . We call a continuous path  $\lambda : [a, b] \rightarrow \text{Homeo}(M)$  a *topological Hamiltonian path* if it satisfies the same conditions with  $\mathbb{R}$  replaced by  $[a, b]$ , and the limit  $L^{(1,\infty)}$ -function  $H_\infty$  called a  $L^{(1,\infty)}$  *topological Hamiltonian* or just a *topological Hamiltonian*.

Following the notations from [OM], we denote by  $\text{Sympeo}(M, \omega)$  the closure of  $\text{Symp}(M, \omega)$  in  $\text{Homeo}(M)$  with respect to the  $C^0$ -metric  $\bar{d}$ , and by  $\mathcal{H}_m([0, 1] \times M, \mathbb{R})$  the set of mean-normalized topological Hamiltonians, and by

$$ev_1 : \mathcal{P}_{[0,1]}^{\text{ham}}(\text{Sympeo}(M, \omega), id) \rightarrow \text{Sympeo}(M, \omega), id \quad (1.4)$$

the evaluation map defined by  $ev_1(\lambda) = \lambda(1)$ . By the uniqueness theorem of Buhovsky-Seyfaddini [BS] (see also [V2] for the  $C^0$ -context), we can extend the map  $\text{Dev}$  given in (1.1) to

$$\overline{\text{Dev}} : \mathcal{P}_{[0,1]}^{\text{ham}}(\text{Sympeo}(M, \omega), id) \rightarrow \mathcal{H}_m([0, 1] \times M, \mathbb{R})$$

in an obvious way. Following the notation of [OM, Oh7], we denote the topological Hamiltonian path  $\lambda = \phi_H$  when  $\overline{\text{Dev}}(\lambda) = \underline{H}$  in this general context.

**Definition 1.3** (Hamiltonian homeomorphism group). We define

$$\text{Hameo}(M, \omega) = ev_1 \left( \mathcal{P}_{[0,1]}^{\text{ham}}(\text{Sympeo}(M, \omega), id) \right)$$

and call any element therein a *Hamiltonian homeomorphisms*.

The group property and its normality in  $Sympeo(M, \omega)$  are proved in [OM].

In [OM], only the (strong) Hamiltonian topology given in Definition 1.1 is studied except at Remark 3.27 [OM]. It appears that the weak Hamiltonian topology, which is induced by the metric on the path space  $\mathcal{P}_{[0,1]}^{ham}(Sympeo(M, \omega), id)$

$$d_{ham}^{weak}(\lambda, \mu) := d_{C^0}(\lambda(1), \mu(1)) + \text{leng}(\lambda^{-1}\mu), \quad (1.5)$$

seems to also play some significant role in the study of  $C^0$  symplectic topology in relation to Lagrangian submanifolds especially *on the cotangent bundle*.

All the above definitions can be modified to handle the case of open manifolds, either noncompact or compact with boundary, by considering compactly supported  $H$ 's as done in section 6 [OM]. Our main interest of noncompact case is the cotangent bundle  $T^*N$  where  $N$  is a closed manifold.

**1.2. Lagrangian spectral invariants.** Let  $N$  be a compact manifold without boundary and let  $T^*N$  be its cotangent bundle equipped with  $\theta$  the Liouville one-form defined by

$$\theta_x(\xi_x) = p(d\pi(\xi_x)), \quad x = (q, p) \in T^*N.$$

The canonical symplectic form  $\omega_0$  on  $T^*N$  is defined by

$$\omega_0 = -d\theta = \sum_{k=1}^n dq^k \wedge dp_k \quad (1.6)$$

where  $(q^1, \dots, q^n, p_1, \dots, p_n)$  is the canonical coordinates of  $T^*N$  associated to the coordinates  $(q^1, \dots, q^n)$  of  $N$ . We put a density  $\rho_N$  on  $o_N$  (or a volume form when  $N$  is oriented), i.e., consider  $o_N$  as a *weighted Lagrangian submanifold*  $(o_N, \rho_N)$  in the sense of Weinstein [W].

Consider Hamiltonian  $H = H(t, x)$  such that  $H_t$  is asymptotically constant, i.e., the ones whose Hamiltonian vector field  $X_H$  is compactly supported. We define

$$\text{supp}_{asc} H = \text{supp} X_H := \bigcup_{t \in [0,1]} X_{H_t}.$$

For each given compact set  $K \subset T^*N$  and  $R \in \mathbb{R}_+$ , we define

$$\mathcal{PC}_{R,K}^\infty = \{H \in C^\infty([0,1] \times T^*N, \mathbb{R}) \mid \text{supp}_{asc} H \subset D^R(T^*N), \|H\| \leq K\} \quad (1.7)$$

which provides a natural filtration of the space  $C^\infty([0,1] \times T^*N, \mathbb{R})$ . We also denote

$$\mathcal{PC}_R^\infty = \bigcup_{K \in \mathbb{R}_+} \mathcal{PC}_{K,R}^\infty, \quad \mathcal{PC}_{asc}^\infty = \bigcup_{R \geq 0} \mathcal{PC}_R^\infty. \quad (1.8)$$

By definition, each element  $H_t$  is independent of  $x = (q, p)$  if  $|p|$  is sufficiently large and so carries a smooth function  $c_\infty : [0,1] \rightarrow \mathbb{R}$  defined by

$$c_\infty(t) = H(t, \infty).$$

Therefore we have the natural evaluation map

$$\pi_\infty : \mathcal{PC}_{asc}^\infty \rightarrow C^\infty([0,1], \mathbb{R}).$$

For each given smooth function  $c : [0,1] \rightarrow \mathbb{R}$ , we denote

$$\mathcal{PC}_{asc;c}^\infty := \pi_\infty^{-1}(c). \quad (1.9)$$

We then introduce the space of Hamiltonian deformations of the zero section and denote

$$\mathfrak{Iso}(o_N; T^*N) = \{\phi_H^1(o_N) \mid H \in \mathcal{PC}_{asc}^\infty\}$$

following the terminology of [W], and

$$\mathfrak{Iso}(o_N; D^R(T^*N)) := \{\phi_H^1(o_N) \mid H \in \mathcal{PC}_R^\infty\}.$$

**Definition 1.4.** We define the *Hamiltonian topology* on  $\mathfrak{Iso}(o_N; D^R(T^*N))$  as the quotient topology of the weak Hamiltonian topology of  $\mathcal{P}^{ham}(Sym_{D^R}(T^*N, \omega), id)$  under the surjective map  $\phi_H \mapsto \phi_H^1(L_0)$  where  $D^R = D^R(T^*N)$ . Then we equip

$$\mathfrak{Iso}(o_N; T^*N) = \lim_{R \rightarrow \infty} \mathfrak{Iso}(o_N; D^R(T^*N))$$

with the direct limit topology of the Hamiltonian topology of  $\mathfrak{Iso}(o_N; D^R(T^*N))$ .

For any given time-dependent Hamiltonian  $H = H(t, x)$ , the classical action functional on the space

$$\mathcal{P}(T^*N) := C^\infty([0, 1], T^*N)$$

is defined by

$$\mathcal{A}_H^{cl}(\gamma) = \int \gamma^* \theta - \int_0^1 H(t, \gamma(t)) dt.$$

We define the subset  $\mathcal{P}(T^*N; o_N)$  by

$$\mathcal{P}(T^*N; o_N) = \{\gamma : [0, 1] \rightarrow T^*N \mid \gamma(0) \in o_N\}.$$

The assignment  $\gamma \mapsto \pi(\gamma(1))$  defines a fibration

$$\mathcal{P}(T^*N; o_N) \rightarrow o_N \cong N$$

with fiber at  $q \in N$  given by

$$\mathcal{P}(T^*N; o_N, T_q^*N) := \{\gamma : [0, 1] \rightarrow T^*N \mid \gamma(0) \in o_N, \gamma(1) \in T_q^*N\}.$$

For given  $x \in L_H$ , we denote the Hamiltonian trajectory

$$z_x^H(t) = \phi_H^t((\phi_H^1)^{-1}(x))$$

which is a Hamiltonian trajectory such that, by definition,

$$z_x^H(0) \in o_N, \quad z_x^H(1) = x. \quad (1.10)$$

We denote  $L_H = \phi_H^1(o_N)$  and by  $i_H : L_H \hookrightarrow T^*N$  the inclusion map.

Motivated by Weinstein's observation that the action functional

$$\mathcal{A}_H^{cl} : \mathcal{P}(T^*N; o_N) \rightarrow \mathbb{R}$$

can be interpreted as the canonical generating function of  $L_H$ , the present author constructed a family of spectral invariants of  $L_H$  by performing a mini-max theory via the chain level Floer homology theory in [Oh2, Oh3]. Indeed, the function defined by

$$h_H(x) = \mathcal{A}_H^{cl}(z_x^H) \quad (1.11)$$

is a canonical generating function of  $L_H$  in that

$$i_H^* \theta = dh_H. \quad (1.12)$$

We call  $h_H$  the *basic generating function* of  $L_H$ . As a function on  $N$ , not on  $L_H$ , it is a multi-valued function. Similarly, one may regard  $N \rightarrow \phi_H^1(o_N)$  as a multi-valued section of  $T^*N$ .

By considering the moduli space of solutions of the perturbed Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J \left( \frac{\partial u}{\partial t} - X_H(u) \right) = 0 \\ u(\tau, 0), u(\tau, 1) \in o_N, \end{cases} \quad (1.13)$$

and applying a chain-level Floer mini-max theory, the author [Oh3] defined a homologically essential critical value, denoted by  $\rho(H; a)$  associated to each cohomology class  $a \in H^*(N)$ . (A similar construction using the generating function method was earlier given by Viterbo [V1] and it is shown in [M, MO] that both invariants coincide *modulo a normalization constant*.) The number  $\rho(H; a)$  depends on  $H$ , not just on  $L_H = \phi_H^1(o_N)$

**1.3. Statement of main results.** The following Hamiltonian continuity result is the Lagrangian analog to Corollary 1.2 of [Sey]. Denote the maximum  $C^0$ -oscillation of  $o_N$  under the Hamiltonian diffeomorphism  $\phi$  by

$$\text{osc}_{C^0}(\phi; o_N) := \max \left\{ \max_{x \in o_N} d(\phi(x), x), \max_{x \in o_N} d(\phi^{-1}(x), x) \right\}. \quad (1.14)$$

**Theorem 1.1** (Theorem 9.1). *Let  $(M, \omega)$  be an arbitrary closed symplectic manifold. Let  $\lambda_i = \phi_{F_i}$  where  $F_i = F_i(t, x)$  be a sequence of smooth Hamiltonians such that*

- (1) *there exists  $R > 0$  such that  $\text{supp } X_{H_i} \subset D^R(T^*N)$  for all  $i$  and  $s \in [0, 1]$ .*
- (2) *There exists a closed ball  $B \subset N$  such that  $\text{supp } \phi_{F_i} \cap o_B = \emptyset$  for all  $i$ 's.*
- (3) *There exists a tubular neighborhood  $T \supset o_B$  such that  $\phi_{H_i}^1 \equiv \text{id}$  on  $T$  for all  $i$ 's.*
- (4)  $\text{osc}_{C^0}(\phi_{F_i}^1; o_N) \rightarrow 0$  as  $i \rightarrow \infty$ .

*Then  $\lim_{i \rightarrow \infty} (\rho(F_i; 1) - \rho(F_i; [pt]^\#)) = 0$ .*

We would like to remark that the condition (3) above automatically satisfies for the Lagrangianization  $\text{Graph } \phi_{F_i}^1$  of the sequence of Hamiltonians  $F_i$  since  $\text{Graph } \phi_{F_i}^1 = \phi_{F_i \oplus 0}(\Delta)$  and  $\phi_{F_i \oplus 0} \equiv \text{id}$  on  $B \times M$  if  $\phi_{F_i} \equiv \text{id}$  on  $B$ .

It turns out that the differences of two spectral invariants like  $\rho^{\text{lag}}(F; 1) - \rho^{\text{lag}}(F; [pt]^\#)$  do not depend on the choice of normalization. Therefore we can define

$$\gamma(L; o_N) := \rho^{\text{lag}}(F; 1) - \rho(F; [pt]^\#)$$

unambiguously which we call the *spectral capacity* of  $L$  (relative to the zero section  $o_N$ ). (See [V1], [Oh3].) We would like to emphasize that a priori it is possible that both  $\rho^{\text{lag}}(F; 1)$  and  $\rho^{\text{lag}}(F; [pt]^\#)$  can have the same sign. This phenomenon is quite a nuisance when one handles the spectral numbers themselves. Because of this, this theorem itself does not tell much about the individual number  $\rho^{\text{lag}}(F_i; 1)$  e.g., it does not imply  $\lim_{i \rightarrow \infty} \rho^{\text{lag}}(F_i; 1) = 0$ .

To properly handle the individual number  $\rho^{\text{lag}}(F; 1)$  and relate it to the Lagrangian submanifold  $L_F = \phi_F^1(o_N)$  itself, not just to  $F$ , we need to put an additional normalization condition relative to  $L_F$ . In this regard, it is useful to take the point of view of weighted Lagrangian submanifolds  $(L, \rho_N)$  introduced in [W], where  $\rho_N$  is a probability density on  $N$ . Using this  $\rho_N$ , we can put a normalization condition with respect to the chosen measure which is the Lagrangian analog to the mean-normalization of Hamiltonians

$$\int_M F(t, x) \omega^n = 0.$$

The next result concerns an enhancement of the construction of basic phase function  $f_H$  carried out in [Oh2] in the level of topological Lagrangian embedding. This

is a graph selector constructed via Lagrangian Floer homology. Then the map  $\sigma_F : N \setminus \text{Sing}(\sigma_F) \rightarrow T^*N$  defined by

$$\sigma_F(q) := df_F(q)$$

selects a single valued branch of  $\phi_F^1(o_N)$  on the open subset  $N \setminus \text{Sing}(\sigma_F)$  of full measure when we regard  $\phi_F^1(o_N)$  as a multi-valued section  $N \rightarrow T^*N$ . We call  $\sigma_F$  *basic Lagrangian selector* of  $\phi_F^1(o_N)$ . In turn the pair  $(\sigma_F, f_F)$  selects a single valued branch of the wave front of  $\phi_F^1(o_N)$  which lies in the one-jet space  $J^1(N) \cong T^*N \times \mathbb{R}$ .

**Theorem 1.2.** *Suppose  $F_i \rightarrow F$  in Hamiltonian topology and denote  $L_i = \phi_{F_i}^1(o_N)$ . Then  $(\sigma_{F_i}, f_{F_i})$  converges uniformly in  $J^1(N)$ , whose limit defines a single-valued continuous section of  $J^1(M)$  on  $N \setminus \text{Sing}(\sigma_F)$ .*

Here we define

$$\text{Sing}(\sigma_F) := \{q \in N \mid f_H \text{ is not differentiable at } q\}$$

and call it the *singular locus* of  $f_H$ . It follows from definition that  $\text{Sing}(\sigma_F)$  is a subset of the so called *Maxwell set* of the Lagrangian projection  $\phi_F^1(o_N) \rightarrow N$ . (See [G1, A3, ZR] for detailed study of the Maxwell set.) We first note that for a generic choice of  $F$ ,  $\text{Sing}(\sigma_F)$  is decomposed into the union of smooth manifolds

$$\text{Sing}(\sigma_F) = \bigcup_{k=1}^n S_k(\sigma_F)$$

where  $S_k(\sigma_F)$  is the stratum of codimension  $k$  in  $N$ . Along each connected component of the codimension one strata  $S_1(\sigma_F)$ ,  $\Sigma_F$  has two branches. We denote by  $f_F^\pm$  the restrictions of  $f_F$  in a neighborhood of the component in each branch respectively.

The next theorem concerns the structure of  $\text{Sing}(\sigma_F)$  in the micro-local level.

**Theorem 1.3** (Theorem 6.1). *Let  $q \in S_1(F)$ . Then*

$$df_F^-(q) - df_F^+(q) \in T_q^*N,$$

*which is contained in the conormal space  $\nu_q^*[S_1(\sigma_F); N] \subset T_q^*N$ .*

In dimension 2, we can precisely define the notion of *cliff-wall surgery*, which replaces the multi-valued graph  $\phi_F^1(o_N)$  by a rectifiable Lagrangian cycle. A finer structure theorem is needed to perform similar surgery in higher dimension which will be studied elsewhere. It appears to the author that these results seem to carry some significance in relation to  $C^0$ -symplectic topology and Hamiltonian dynamics, which may be worthwhile to pursue further in the future.

Finally we prove the following inequality between the basic phase function and the Lagrangian spectral invariants.

**Theorem 1.4** (Theorem 8.1). *For any  $F$ , we have*

$$\max f_F \leq \rho^{lag}(F; 1).$$

Its proof uses a judicious usage of the triangle product in Lagrangian Floer homology [Oh3, Se, FOOO1] after a careful consideration of normalization problem in section 7.4. We would like to emphasize that the issue of normalization problem concerning  $\rho^{lag}(F; 1)$  is a delicate one when one would like to regard  $\rho^{lag}(F; 1)$  as an invariant attached to the Lagrangian submanifold itself, not just to the Hamiltonian  $F$ .

The research performed in this paper is partially motivated by the study of topological Hamiltonian dynamics and its applications to the problem of simpleness question on the area-preserving homeomorphism group of the 2-disc. It should also be regarded as a natural continuation of the author's study of Lagrangian spectral invariants performed in [Oh2].

We thank F. Zapolsky for attracting our attention to the preprint [MVZ] from which we have learned the Lagrangian version of the optimal triangle inequality, and S. Seyfaddini for sending us his very interesting preprint [Sey], which greatly helps us in proving the Hamiltonian continuity of Lagrangian spectral capacity. We also thank A. Givental for many enlightening e-mail communications concerning the structure of Maxwell set, Proposition 6.2 and the cliff-wall surgery.

### Notations and Conventions

We follow the conventions of [Oh6, Oh7] for the definition of Hamiltonian vector fields and action functional, and others appearing in the Hamiltonian Floer theory and in the construction of spectral invariants on general closed symplectic manifold. They are different from e.g., those used in [Po, EP] one way or the other, but coincide with those used in [Sey].

- (1) We usually use the letter  $M$  to denote a symplectic manifold and  $N$  to denote a general smooth manifold.
- (2) The Hamiltonian vector field  $X_H$  is defined by  $dH = \omega(X_H, \cdot)$ .
- (3) The flow of  $X_H$  is denoted by  $\phi_H : t \mapsto \phi_H^t$  and its time-one map by  $\phi_H^1 \in \text{Ham}(M, \omega)$ .
- (4) We denote by  $z_H^q(t) = \phi_H^t(q)$  the Hamiltonian trajectory associated to the initial point  $q$ .
- (5) We denote by  $z_x^H(t) = \phi_H^t((\phi_H^1)^{-1}(x))$  the Hamiltonian trajectory associated to the final point  $x$ .
- (6)  $\overline{H}(t, x) = -H(t, \phi_H^t(x))$  is the Hamiltonian generating the inverse path  $(\phi_H^t)^{-1}$ .
- (7) The canonical symplectic form on the cotangent bundle  $T^*N$  is denoted by  $\omega_0 = -d\theta$  where  $\theta$  is the Liouville one-form which is given by  $\theta = \sum_i p_i dq^i$  in the canonical coordinates  $(q^1, \dots, q^n, p_1, \dots, p_n)$ .
- (8) The classical Hamilton's action functional on the space of paths in  $T^*N$  is given by

$$\mathcal{A}_H^{\text{cl}}(\gamma) = \int \gamma^* \theta - \int_0^1 H(t, \gamma(t)) dt.$$

- (9) We denote by  $o_N$  the zero section of  $T^*N$ .
- (10) We denote  $\rho^{\text{lag}}(H; a)$  the Lagrangian spectral invariant on  $T^*N$  (relative to the zero section  $o_N$ ) defined in [Oh2] for asymptotically constant Hamiltonian  $H$  on  $T^*N$ .
- (11) We denote by  $f_H$  the basic phase function and its associated Lagrangian selector by  $\sigma_H : N \rightarrow T^*N$  given by  $\sigma_H(q) = df_H(q)$  at which  $df_H(q)$  exists.
- (12)  $\varphi^H = (\phi_H^1)^{-1} \circ \sigma_H : o_N \rightarrow o_N$  is the measurable map associated to the Lagrangian selector  $\sigma_H$ .
- (13)  $\Sigma_H = \text{Im } \sigma_H \subset \phi_H^1(o_N)$  and  $U(\Sigma_H) = (\phi_H^1)^{-1}(\Sigma_H) \subset o_N$ .
- (14)  $\sigma_F^{\text{add}}$  is the basic Lagrangian cycle.



2. BASIC GENERATING FUNCTION  $h_H$  OF LAGRANGIAN SUBMANIFOLD

In this section, we recall the definition of *basic generating function*.

Let  $H = H(t, x)$  be a Hamiltonian on  $T^*N$  which is *asymptotically constant* i.e., one whose Hamiltonian vector field  $X_H$  is compactly supported. Denote by  $\mathcal{PC}_{asc}^\infty(T^*N, \mathbb{R})$  be the set of such a family of functions. We denote  $L_H = \phi_H^1(o_N)$  and denote by  $i_H : L_H \hookrightarrow T^*N$  the inclusion map.

Recall the classical action functional is defined as

$$\mathcal{A}_H^{cl}(\gamma) = \int \gamma^* \theta - \int_0^1 H(t, \gamma(t)) dt$$

on the space  $\mathcal{P}(T^*N)$  of paths  $\gamma : [0, 1] \rightarrow T^*N$ , and its first variation formula is given by

$$d\mathcal{A}_H^{cl}(\gamma)(\xi) = \int_0^1 \omega(\dot{\gamma} - X_H(t, \gamma(t)), \xi(t)) dt - \langle \theta(\gamma(0)), \xi(0) \rangle + \langle \theta(\gamma(1)), \xi(1) \rangle. \quad (2.1)$$

For given  $q \in o_N \cong N$ , we denote

$$z_H^q(t) = \phi_H^t(q)$$

which is a Hamiltonian trajectory such that

$$z_H^q(0) = q \in o_N, \quad (2.2)$$

which specifies the *initial point*  $q \in o_N$ . (We remark that the notation here is slightly different from that of [Oh2, Oh3] in that  $z_H^q$  therein denotes  $z_q^H$  in this paper. We adopt the current notation to be consistent with that of [Oh8] and other recent papers of the author.)

We define the function  $\tilde{h}_H : [0, 1] \times N \rightarrow \mathbb{R}$  by

$$\tilde{h}_H(t, q) = \int (z_H^q|_{[0, t]})^* \theta - \int_0^t H(u, \phi_H^u(q)) du \quad (2.3)$$

call it the space-time (or parametric) *basic generating function* in the fixed frame.

The following basic lemma follows immediately from (4.4) whose proof we omit.

**Lemma 2.1.** *The function  $\tilde{h}_H$  satisfies*

$$d\tilde{h}_H(t, q) = ((z_H^q)^* \theta(t) - H(t, z_H^q(t)) dt) + (\psi_H^t)^* \theta \quad (2.4)$$

$$= \psi_H^* \theta - H(t, z_H^q(t)) dt \quad (2.5)$$

where  $\psi_H : [0, 1] \times N \rightarrow T^*N$  defined by  $\psi_H(t, q) = \phi_H^t|_{o_N}$  and  $\psi_H^t(q) = \psi_H(t, q)$ .

It turns out that the following form of Hamiltonian trajectories

$$z_x^H(t) = \phi_H^t((\phi_H^1)^{-1}(x)) \quad (2.6)$$

are also useful, which specifies the *final point* of the trajectory instead of the initial point as specified in the trajectory  $z_H^q$ . Then we define

$$h_H(t, x) = \tilde{h}_H(t, (\phi_H^t)^{-1}(x)), \quad x \in \phi_H^t(o_N) \quad (2.7)$$

in the *moving frame*.

Now consider the Lagrangian submanifold  $\phi_H^1(o_N)$ . We would like to point out that the function

$$h_H(1, \cdot) : L_H \rightarrow \mathbb{R}; \quad h_H(1, x) := \tilde{h}_H(1, (\phi_H)^{-1}(x))$$

defines the natural generating function of  $L_H := \phi_H^1(o_N)$  in that  $d_x h_H = i_H^* \theta$  where  $i_H : L_H \rightarrow T^*N$  is the canonical inclusion map. The image of the map

$$x \in L_H \mapsto (h_H(x), x)$$

defines a canonical Legendrian lift of  $L_H$  in the one-jet bundle  $J^1(N) \cong \mathbb{R} \times T^*N$ . We call  $h_H$  the *basic generating function in the moving frame*. We denote the corresponding Legendrian submanifold by  $R_H$ . However, as a function on  $N$ ,  $h_H$  is multi-valued, while  $\tilde{h}_H$  is a well-defined single-valued function. (We refer to section 3.2 for further discussion on the space-time (or parametric) basic generating function  $h_H$ .)

In general, the projection  $R \rightarrow \mathbb{R} \times N$  of any Legendrian submanifold  $R \subset J^1(N, \mathbb{R}) = \mathbb{R} \times T^*N$  is called the *wave front* [El] of the Legendrian submanifold  $R$ . We denote by  $W_R \subset \mathbb{R} \times N$  by the front of  $R$ . We also define the (Lagrangian) action spectrum of  $H$  on  $T^*N$  by

$$\text{Spec}(H; N) = \{\mathcal{A}_H^{\text{cl}}(z_x^H) \mid x \in L_H \cap o_N\} \quad (2.8)$$

which also coincides with the set of critical values of  $h_H$ . It follows that  $\text{Spec}(H; N)$  is a compact subset of  $\mathbb{R}$  of measure zero.

**Remark 2.1.** We would like to note that we have no a priori control of  $C^0$  bound for the functions  $h_H$  (or equivalently  $\tilde{h}_H$ ), even when  $H$  is bounded in  $L^{(1, \infty)}$  norm. Getting this  $C^0$ -bound is equivalent to getting the bound for the actions of the relevant Hamiltonian chords. Indeed understanding the precise relationship between the action bound, the norm  $\|H\|$  and the  $C^0$ -distance of the time-one map  $\phi_H^1$  is a heart of the matter in  $C^0$  symplectic topology.

In section 5, we recall construction of *basic phase function*  $f_H$  from [Oh2] which is a particular single valued selection of the multivalued function  $h_H$  on  $N$  that has particularly nice properties in relation to the study of spectral invariants of the present paper. This function was constructed via the Floer mini-max arguments similarly as the spectral invariants  $\rho^{\text{ham}}(H; a)$  is defined in [Oh2], and its  $C_0$ -norm is bounded by  $\|H\|$ .

### 3. MULTI-TIME EXTENDED PHASE SPACE AND LAGRANGIAN SUSPENSION

In this section, we generalize construction of Lagrangian suspension in the setting of multi-time extended phase space. For the simplicity of exposition, we will mainly consider the 2-parameter family whose generalization to multi-parameter case is obvious and so omitted.

**3.1. Lagrangian suspension on general  $(M, \omega)$ .** Generalizing our notations  $z_F^q$  and  $z_x^F$  we introduced in section 4 for the multi-parameter families, we associate several maps to a given parameter family

$$\Lambda = \{\phi(s, t); \quad \phi(s, t)\}. \quad (3.1)$$

**Remark 3.1.** A natural 2-parameter family occurs in the form of  $\phi(s, t) = \phi_{H(s)}^t$  when a 2-parameter family  $H = H(s, t, x)$  is considered. One should regard this particular two-parameter family of Hamiltonian diffeomorphisms as the *one-parameter family* of Hamiltonian paths

$$[0, 1] \rightarrow \mathcal{P}^{\text{ham}}(\text{Symp}(M, \omega), \text{id}); \quad s \mapsto \phi_{H(s)}$$

instead of a two-parameter family of Hamiltonian diffeomorphisms in  $Ham(M, \omega)$ . This is because for a given general two parameter family  $\{\phi(s, t)\}$ , its  $t$ -Hamiltonian is not a priori given but should be computed unlike the family generated by the given Hamiltonians  $\mathbb{H}(s)$  in (3.1).

**Definition 3.2.**

$$\begin{aligned} z_\Lambda : [0, 1]^2 \times M &\rightarrow M; & z_\Lambda(s, t, q) &= \Lambda(s, t)(q) \\ z^\Lambda : [0, 1]^2 \times M &\rightarrow M; & z^\Lambda(s, t, x) &= \Lambda(s, t)\Lambda(s, 1)^{-1}(x) \end{aligned}$$

and

$$\begin{aligned} \phi_\Lambda : [0, 1]^2 \times M &\rightarrow [0, 1]^2 \times M; & \psi_\Lambda(s, t, q) &= (s, t, z_\Lambda(s, t, q)) \\ \psi_\Lambda : [0, 1]^2 \times M &\rightarrow [0, 1]^2 \times M; & \psi_\Lambda(s, t, x) &= (s, t, z^\Lambda(s, t, x)). \end{aligned}$$

Now suppose that  $g_{s,t} : L \rightarrow (M, \omega)$  is a 2-parameter family of Lagrangian embeddings generated by a 2-parameter family of Hamiltonian diffeomorphisms  $\{\phi(s, t)\}$  with  $\phi(0, 0) = id$ , i.e., satisfies

$$g_{(s,t)}(q) = \phi(s, t)(g_{(0,0)}(q)), \quad q \in L.$$

There are two different ways of describing this family of embeddings. One is in terms of the embedding maps  $g_{s,t}$  and the other as a family of moving submanifolds  $L_{(s,t)} = \text{Im } g_{(s,t)} \subset M$ . One may regard the first picture as the one in the ‘fixed frame’ and the second as the one in the ‘moving frame’. Both of them can be realized by the ambient 2-parameter isotopies with its space-time images coincide which is given by the trace

$$\widehat{L} := \mathbf{Tr}_\Lambda(L) = \phi_\Lambda([0, 1]^2 \times L).$$

In terms of  $\psi_\Lambda$ , we have the following explicit formula

$$\psi_\Lambda(s, t, x) = \phi_\Lambda(s, t, z^\Lambda(s, t, x)), \quad x \in \widehat{L} \quad (3.2)$$

which is the parametric analog to the relationship

$$z_F^{\phi_F^{-1}(x)} = z_x^F$$

between two representations of Hamiltonian trajectories, one in terms of the initial condition and the other in terms of the final condition.

Denote by  $H, G$  the corresponding  $t$ - and  $s$ -Hamiltonians respectively so that

$$g_{(s,t)}(x) = \Lambda(s, t)(x), \quad x \in L.$$

Denote by  $\Phi_\Lambda : [0, 1]^2 \times L \rightarrow T^*[0, 1]^2 \times M$  the associated double-suspension defined by

$$\Psi_\Lambda(s, t, q) = (\Lambda(s, t)(q), s, G(s, t, \Lambda(s, t)(q)), t, -H(s, t, \Lambda(s, t)(q))). \quad (3.3)$$

In the moving frame, it is simply given by

$$\Phi_\Lambda(s, t, x) = (x, s, G(s, t, x), t, -H(s, t, x)) \quad (3.4)$$

on the space-time trace  $\widehat{L} \subset [0, 1]^2 \times M$ .

**Lemma 3.1.** *Let  $z_\Lambda, z^\Lambda, \phi_\Lambda$  and  $\psi^\Lambda$  be as above. Then*

$$(1) \quad z_\Lambda^* \omega = \phi_\Lambda^*(dG) \wedge dt - \phi_\Lambda^*(dH) \wedge ds \quad (3.5)$$

which is also equivalent to saying that the suspension  $\Phi_\Lambda$  is a Lagrangian embedding, i.e., satisfies

$$\Phi_\Lambda^*(\omega + da \wedge dt + db \wedge ds) = 0 \quad (3.6)$$

on  $[0, 1]^2 \times L$ .

(2) In the moving frame,

$$(z^\Lambda)^* \omega = dG \wedge dt - dH \wedge ds \quad (3.7)$$

on the trace  $\widehat{L}$  which is equivalent to saying that the trace  $\widehat{L}$  is a Lagrangian submanifold with respect to the symplectic form

$$\omega + da \wedge dt + db \wedge ds$$

on  $M \times T^*[0, 1]^2$ .

**Remark 3.3.** The expression (3.7) has more natural invariant meaning than (3.5) in that in this representation, it is psychologically easier to allow more general class of subsets than smooth submanifolds such as basic Lagrangian selector or its wave front propagation as the integral domains against smooth differential forms. For this integration purpose, it is important to observe that these wave front type objects define *rectifiable geometric chain residing in  $T^*\Delta$*  in the sense of geometric measure theory [Fe]. In fact, all of the chains we are considering are *integral currents* in that their boundaries are also rectifiable. (See [Fe] section 4.1.1, 4.1.7.)

The following particular homotopy seems to deserve a name for it.

**Definition 3.4** (History homotopy). For any given Hamiltonian  $F = F(t, x)$ , we call the homotopy

$$\mathcal{F} : v \mapsto \phi_{F^v}$$

the (past) *history homotopy* of  $\phi_F^1$ .

We use the corresponding script letter to denote this homotopy for each given Hamiltonian. Their  $t$ -time-one maps are nothing but  $\phi_F^v$  for each  $v \in [0, 1]$ .

**3.2. Lagrangian suspension on the cotangent bundle.** Now we specialize the suspension construction to the case of cotangent bundle. We will show that in this case we can go one step further in that we can also write down the generating function of the Lagrangian suspension when regarded as an exact Lagrangian submanifold in the cotangent bundle  $T^*\Delta \times T^*[0, 1]^2 \cong T^*(\Delta \times [0, 1]^2)$ .

Let  $\{F\}$  be a time-dependent Hamiltonians defined on the cotangent bundle  $T^*N$ . Then we define the parametric version of basic generating functions

$$\widetilde{h}_F : [0, 1] \times \Delta \rightarrow \mathbb{R}; \quad \widetilde{\mathbf{h}}(v, q) = \widetilde{h}_{F^v}(1, q) = \widetilde{h}_F(v, q)$$

where the basic space-time generating function associated to  $F$  was defined in (2.3) as

$$\widetilde{h}(v, q) = \int_0^v (z_F^q|_{[0, v]})^* \theta - \int_0^v F(u, z_F^q(t)) dt$$

with substitution of  $v = t$ , and  $h_H$  the one defined by

$$h_F(t, x) = \widetilde{h}_F(t, (\phi_{F^t}^1)^{-1}(x)) = h_{F^t}(1, x) \quad (3.8)$$

which are defined on  $\widehat{L}$ . We call  $\widetilde{h}_F$  the (space-time) basic generating function *in the fixed frame* and  $h_F$  that *in the moving frame*. Note that the definitions of both  $\widetilde{h}_F$  and  $h_F$  involve only the time  $t = 1$  for the family  $\mathcal{F} = \{\phi_{F^v}^t\}_{v \in [0,1]}$ , which is consistent with the fact that they depend only on the time-one image  $\phi_{F^v}^1(o_N) (= \phi_F^v(o_N))$  of the zero section (under a suitable support hypothesis on the Hamiltonians such as given in the present paper).

With this remark made, we switch the parameter  $v$  by  $t$  or substitute  $v = t$ , and derive the following proposition by applying the history homotopy to  $H(s)$  for each  $s \in [0, 1]$ .

**Proposition 3.2.** *Assume that  $\{H(s)\}$  is a 1-parameter family of  $t$ -Hamiltonians and denote by  $G = G(s, t, x)$  its  $s$ -Hamiltonian. Assume both  $H$  and  $G$  are boundary flat, i.e., satisfy  $G \equiv 0$  near  $s = 0$  and  $H \equiv 0$  near  $t = 0$  respectively. Then the basic generating function  $\widetilde{\mathbf{h}}$  satisfies*

$$\begin{aligned} \frac{\partial \widetilde{\mathbf{h}}}{\partial t} &= (\phi_H^t)^*(-H + \langle \Theta, X_H \rangle) \\ \frac{\partial \widetilde{\mathbf{h}}}{\partial s} &= (\phi_H^t)^*(G - \langle \Theta, X_G \rangle) \end{aligned} \quad (3.9)$$

for all  $(s, t, q) \in [0, 1]^2 \times \Delta$ , and  $\mathbf{h}$  is a generating function of the Lagrangian submanifold  $\widehat{L}$ , i.e., satisfies

$$d\mathbf{h} = i_{\widehat{L}}^*(\Theta + a dt + b ds) \quad (3.10)$$

where  $i_{\widehat{L}} : \widehat{L} \rightarrow T^*\Delta \times T^*[0, 1]^2$  is the inclusion map.

*Proof.* Recall the identity (3.9) which can be also written as

$$dh_F(t, x) = \psi_F^*(\theta - F dt) \quad \text{for } x \in \phi_F^t(o_N).$$

Applying this to each  $F = H(s)$  in the current case and taking the interior product with  $\frac{\partial}{\partial t}$ , we obtain the first equality of (3.9). The second equation can be derived by changing the role of  $s$  and  $t$  and using the equality of  $\widetilde{\mathbf{h}}_{H(s)}(t, x) = \widetilde{h}_G(s, x)$ . The equation (3.10) follow from the definition  $\mathbf{h}(s, t, x) = \widetilde{\mathbf{h}}(t, (\phi_{H(s)}^t)^{-1}(x))$  and (3.9) whose derivation is left to the readers.  $\square$

It is often more convenient to say that the equation (3.10) holds on  $\widehat{L}$  instead of using the formal expression when one allows more general class of subsets like basic Lagrangian selectors as the integration domain. In particular, we may simply say

$$\frac{\partial \mathbf{h}}{\partial t} = -H, \quad \frac{\partial \mathbf{h}}{\partial s} = G, \quad d_{T^*N} \mathbf{h} = \Theta \quad (3.11)$$

on the smooth locus of  $\widehat{L}$  in the moving frame, when  $\widehat{L}$  is not smooth everywhere.

**Remark 3.5.** (1) The same kind of equation holds for a multi-time family which we will not discuss in this paper, since we do not need it.  
 (2) Note that the domain of  $\mathbf{h}$  is  $\widehat{L}$ , a subset of  $[0, 1]^2 \times T^*\Delta$ , not the whole space.

The extended phase space  $\mathbb{R}_t \times T^*N$  has its companion, called *one-jet space*  $J^1(N)$  which itself has the form  $\mathbb{R}_z \times T^*N$ . The space carries a natural *contact*

structure induced by the contact form  $dz - p dq$ . The combined space  $\mathbb{R}_t \times \mathbb{R}_z \times T^*N$  can be formed into  $T^*(\mathbb{R} \times N)$  with *symplectic form*

$$dt \wedge dz - d\Theta = dt \wedge dz + \sum_i dq_i \wedge dp_i.$$

The pair  $(h_H, L_H) \subset \mathbb{R}_z \times T^*N \cong J^1(N)$  forms a *Legendrian submanifold* at each time  $t$ , and its time propagation is called the *wave front propagation*. We call the image  $W_H$  of the front projection  $\mathbb{R}_z \times T^*N \rightarrow \mathbb{R}_z \times N$  the *front propagation*.

#### 4. CANONICAL FORMALISM AND GENERATING FUNCTION

In this section, we interpret the materials developed in the previous two sections into the well-established language of canonical formalism in Hamiltonian mechanics in the extended phase space. (See e.g., [Go], [A2].)

We review Arnold's description of canonical formalism in the 'extended phase space'  $\mathbb{R} \times T^*\Delta$  (see section 44, 45 [A2]) in this general cotangent bundle setting. Consider the product

$$(\mathbb{R} \times T^*X) \times (\mathbb{R} \times T^*X)$$

and denote by  $(t, x) = (t, (q, p))$  the coordinates of the first factor, the initial point, and  $(T, X) = (T, (Q, P))$  the coordinates of the second factor, the final point of a given Hamiltonian trajectory. We consider the classical action functional in the extended space time or in the path-space in the 'extended space time'.

**Definition 4.1** (Extended path-space). Define  $\mathcal{P}^{ext}(T^*X)$  to be the union

$$\mathcal{P}^{ext}(T^*N) = \bigcup_{(v, T) \in \mathbb{R}^2} \{(v, T)\} \times \{\gamma \in C^\infty([v, T]; T^*X)\} \quad (4.1)$$

By definition, we have natural evaluation maps

$$ev_{ini}, ev_{fin} : \mathcal{P}^{ext}(T^*N) \rightarrow \mathbb{R} \times T^*N$$

defined by

$$ev_{ini}(\gamma) = (v, \gamma(v)), \quad ev_{fin}(\gamma) = (T, \gamma(T)). \quad (4.2)$$

Now suppose we are given a time-dependent Hamiltonian function  $H : \mathbb{R} \times T^*N \rightarrow \mathbb{R}$ . It naturally introduce the following parametric canonical relation

$$\{((v, x), (T, X)) \mid X = (\phi_H)_v^T(x)\}$$

which in turn provides a natural boundary condition for the paths in the space-time. Here  $(\phi_H)_v^T$  denotes the time  $(v, T)$ -map with initial time  $v$  and final time  $T$ . Recall that the map  $\phi_H$  denotes the time-dependent flow it actually should be regarded as a map

$$\phi_H : \mathbb{R}^2 \times T^*N \rightarrow T^*N$$

defined by

$$\phi_H(v, T, x) = z_H^x(v, T)$$

where  $T \mapsto z_H^x(T)$  is the solution to the initial value problem

$$\dot{x} = X_H(T, x), \quad x(v) = x. \quad (4.3)$$

Now we consider the classical action functional

$$\mathcal{A}_H^{cl}(\gamma) = \int \gamma^* \theta - \int_v^T H(t, \gamma(t)) dt$$

as physicists treat it, i.e., with no specification of domains of  $\gamma$  and with no boundary condition specified. This means that we are considering the action functional on the extended phase space

$$\mathbb{R} \times T^*N$$

i.e.,  $\mathcal{P}^{ext}(T^*N)$ . For  $\gamma \in \mathcal{P}([v, T]; T^*N)$ , the action is defined to be

$$\mathcal{A}_H^{cl}(\gamma) = \int \gamma^* \theta - \int_v^T H(t, \gamma(t)) dt.$$

Its first variation formula for fixed  $v, T$  is given by

$$d\mathcal{A}_H^{cl}(\gamma)(\xi) = \int_v^T \omega(\dot{\gamma} - X_H(t, \gamma(t)), \xi(t)) dt - \langle \theta(\gamma(v)), \xi(v) \rangle + \langle \theta(\gamma(T)), \xi(T) \rangle. \quad (4.4)$$

**Remark 4.2.** Based on this formula, one may regard, as Weinstein put it, *the restriction  $\mathcal{A}_H^{cl}|_{\mathcal{P}_{[0,1]}(T^*N; o_N)}$  is a generating function of  $\phi_H^1(o_N)$  which we regard as a function defined on the fibration*

$$ev_{fin} : \mathcal{P}_{[0,1]}(T^*N; o_N) \rightarrow T^*N.$$

Here is the analog to this proposition on the extended phase space. We denote

$$\mathcal{P}_{[0,T]}^{ext}(T^*N) = \bigcup_{T \in \mathbb{R}} \mathcal{P}_{[0,T]}(T^*N).$$

Denote by  $z_{H;T}^x : [v, T] \rightarrow T^*N$  the Hamiltonian trajectory satisfying the initial condition

$$z_{H;T}^{(v;x)}(v) = x,$$

of  $z_{H;T}^{(v;x)}(v) = (\phi_H)_v^v(x) = x$ . Now we consider the restriction of the action functional  $\mathcal{A}^{cl}$  to

$$\mathcal{P}_{[v,T]}^{ext}(T^*N; o_N) = \{(T, \gamma) \mid \gamma \in C^\infty([v, T]; T^*N), \gamma(v) \in o_N\}.$$

Then we define a function

$$S(v, T, x) = \mathcal{A}^{cl}(z_{H;T}^{(v;x)}) \quad (4.5)$$

**Proposition 4.1** (p. 241 [A2]). *Denote  $X = X(v, T, x) := (\phi_H)_v^T(x)$  and*

$$K(v, T, x) = H(T, (\phi_H)_v^T(x)) = H(T, X).$$

*Then the function  $S : \mathbb{R}^2 \times T^*N \rightarrow \mathbb{R}$  satisfies*

$$p dq - H dt = P dQ - K dT + dS \quad (4.6)$$

*or equivalently*

$$dS = p dq - H dt - (P dQ - K dT).$$

Now we specialize our discussion to the case of  $[v, T] = [0, 1]$ . For given  $q \in o_N \cong N$ , we denote  $z_H^q(t) = \phi_H^t(q)$  which is the Hamiltonian trajectory such that

$$z_H^q(0) = q \in o_N, \quad (4.7)$$

which specifies the *initial point*  $q \in o_N$  of a Hamiltonian trajectory. We also represent the same Hamiltonian trajectory as

$$z_x^H(t) = \phi_H^t((\phi_H^1)^{-1}(x)) \quad (4.8)$$

which specifies the *final point* of the trajectory instead of the initial point as specified for the trajectory  $z_H^q$ .

In the footnote in p.236 of [A2], Arnold made a comment “the form

$$\omega^1 := p dq - H dt = \theta - H dt$$

seems to appear out of thin air” and alluded its origin from optics. In fact, this form is nothing but the pull-back of the canonical one-form  $\theta + a dt$  under the Lagrangian suspension. (See the next section of the present paper.)

In particular, we derive

**Proposition 4.2.** *The graph*

$$\text{Graph } \phi_\Lambda := \{((T, (Q, P)), (t, (q, p))) \mid (Q, P) = (\phi_H)_t^T(q, p)\} \quad (4.9)$$

*is a Lagrangian submanifold of*

$$((T^*N \times T^*\mathbb{R}) \times (T^*N \times T^*\mathbb{R}), \quad \pi_1^*(\omega_0 + dt \wedge da) - \pi_2^*(\omega_0 + dt \wedge da))$$

*and its generating function is given by  $S$ .*

There are two different ways of writing down the generating function  $S$ , one is with respect to the initial condition and the other with respect to the final condition. We call the first point of view that *with respect to fixed frame* and the second *with respect to the moving frame*.

In the fixed frame, or equivalently in terms of  $(t, (q, p))$  (4.6) becomes the equation

$$dS = \Phi_\Lambda^*(\Theta + a dt)$$

where  $\Phi_\Lambda$  is the Lagrangian suspension defined by

$$\Phi_\Lambda(t, \mathbf{q}) = (\phi_{\mathbb{H}}^t(\mathbf{q}), t, -\mathbb{H}(s, t, \phi_{\mathbb{H}}^t(\mathbf{q}))), \quad \mathbf{q} = ((q, p), (q, p)). \quad (4.10)$$

The domain of  $\Phi_\Lambda$  is given by  $\mathbb{R} \times \Delta$  and

$$S(t, \mathbf{q}) = \tilde{h}_H(t, \mathbf{q})$$

where  $\tilde{h}_H(t, \mathbf{q})$  is the function defined by

$$\tilde{h}_H(t, \mathbf{q}) = \mathcal{A}_H^{cl}(z_{\mathbb{H}}^{\mathbf{q}}).$$

In the moving frame, or equivalent in terms of  $(T, (Q, P))$ , (4.6) becomes

$$dS = \Psi_\Lambda^*(\Theta + a dt)$$

where  $\Psi_\Lambda$  is nothing but the inclusion map into  $T^*\Delta \times T^*\mathbb{R}$  with its domain given by

$$\bigcup_{T \in \mathbb{R}} \{T\} \times \phi_H^T(\Delta).$$

Its generating function can be written as

$$S(T, (Q, P)) =: h_H(T, (Q, P))$$



## 5. BASIC PHASE FUNCTION AND ITS ASSOCIATED LAGRANGIAN SELECTOR

In this section, we first recall the definition of *basic phase function* constructed in [Oh2]. Then we introduce a crucial measurable map  $\varphi^H : N \rightarrow N$ , which is defined by a selection of a single valued branch of the multivalued section

$$N \rightarrow L_H \subset T^*M$$

followed by  $(\phi_H^1)^{-1}$ . We call this map the mass transfer map associated to the Hamiltonian  $H$ . It is interesting to note that such a selection process was studied e.g., in the theory of multi-valued functions, or  $Q$ -valued functions, in the sense of Almgren [Al] in geometric measure theory. In particular, in [DGT], existence of such a single valued branch is studied in the general abstract setting of metric spaces and a finite group action of isometries. It would be interesting to see whether there would be any other significant intrusion of the theory of multivalued functions into the study of symplectic topology.

**5.1. Graph selector of wave fronts.** The following theorem was proved in [Cha] and in [Oh2] by the generating function method and by the Floer theory respectively. (According to [PPS], the proof of this theorem was first outlined by Sikorav in Chaperon's seminar.)

**Theorem 5.1** (Sikorav, Chaperon [Cha], Oh [Oh2]). *Let  $L \subset T^*N$  be a Hamiltonian deformation of the zero section  $o_N$ . Then there exists a Lipschitz continuous function  $f : N \rightarrow \mathbb{R}$ , which is smooth on an open subset  $N_0 \subset N$  of full measure, such that*

$$(q, df(q)) \in L$$

*for every  $q \in N_0$ . Moreover if  $df(q) = 0$  for all  $q \in N_0$ , then  $L$  coincides with the zero section  $o_N$ . The choice of  $f$  is unique modulo the shift by a constant.*

The details of the proof of Lipschitz continuity of  $f$  is given in [PPS]. We denote by  $\text{Sing } f$  the set of non-differentiable points of  $f$ . Then by definition

$$N_0 = \text{Reg } f := N \setminus \text{Sing } f$$

is a subset of full measure and  $f$  is differentiable thereon.

We call such a function  $f$  a *graph selector* in general following the terminology of [PPS] and denote the corresponding graph part of the front of the Legendrian submanifold  $R$  by

$$G_f := \{(h_L(q, df(q)), q, df(q)) \mid q \in N\} \subset R.$$

By construction, the projection  $\pi_R : G_f \rightarrow N$  restricts to a one-one correspondence and the function  $f : \text{Reg } f \rightarrow \mathbb{R}$  continuously extends to  $\overline{\text{Reg } f} = N$ .

By definition,

$$|df(q)| \leq \max_{x \in L} |p(x)| \quad (5.1)$$

for any  $q \in N_0$ , where  $x = (q(x), p(x))$  and the norm  $|p(x)|$  is measured by any given Riemannian metric on  $N$ .

**Proposition 5.2.** *As  $d_H(L, o_N) \rightarrow 0$ ,  $|df(q)| \rightarrow 0$  uniformly over  $q \in N_0$ .*

In [Oh2], a canonical choice of  $f$  is constructed via the chain level Floer theory, provided the generating Hamiltonian  $H$  of  $L$  is given. The author called the corresponding graph selector  $f$  the *basic phase function* of  $L = \phi_H^1(o_N)$  and denoted it

by  $f_H$ . We give a quick outline of the construction referring the readers to [Oh2] for the full details of the construction.

**5.2. The basic phase function  $f_H$  and its Lagrangian selector.** Another construction in [Oh2] is given by considering the Lagrangian pair

$$(o_N, T_q^* N), \quad q \in N$$

and its associated Floer complex  $CF(H; o_N, T_q^* N)$  generated by the Hamiltonian trajectory  $z : [0, 1] \rightarrow T^* N$  satisfying

$$\dot{z} = X_H(t, z(t)), \quad z(0) \in o_N, z(1) \in T_q^* N. \quad (5.2)$$

Denote by  $Chord(H; o_N, T_q^* N)$  the set of solutions. The differential  $\partial_{(H, J)}$  on  $CF(H; o_N, T_q^* N)$  is provided by the moduli space of solutions of the perturbed Cauchy-Riemann equation

$$\begin{cases} \frac{\partial u}{\partial \tau} + J \left( \frac{\partial u}{\partial t} - X_H(u) \right) = 0 \\ u(\tau, 0) \in o_N, u(\tau, 1) \in T_q^* N. \end{cases} \quad (5.3)$$

An element  $\alpha \in CF(H; o_N, T_q^* N)$  is expressed as a finite sum

$$\alpha = \sum_{z \in Chord(H; o_N, T_q^* N)} a_z[z], \quad a_z \in \mathbb{Z}.$$

We denote the level of the chain  $\alpha$  by

$$\lambda_H(\alpha) := \max_{z \in \text{supp } \alpha} \{\mathcal{A}_H^{cl}(z)\}.$$

The resulting invariant  $\rho^{lag}(H; \{q\})$  is to be defined by the mini-max value

$$\rho^{lag}(H; \{q\}) = \inf_{\alpha \in [q]} \lambda_H(\alpha)$$

where  $[q] \in H_0(\{q\}; \mathbb{Z})$  is a generator of the homology group  $H_0(\{q\}; \mathbb{Z})$ .

A priori,  $\rho^{lag}(H; \{q\})$  is defined when  $\phi_H^1(o_N)$  intersects  $T_q N^*$  transversely but can be extended to non-transversal  $q$ 's by continuity. By varying  $q \in N$ , this defines a function  $f_H : N \rightarrow \mathbb{R}$  which is precisely the one called the basic phase function in [Oh2]. (A similar construction of such a function using the generating function method was earlier given by Sikorav and Chaperon [Cha].) We call the associated graph part  $G_{f_H}$  the *basic branch* of the front  $W_{R_H}$  of  $R_H$ .

**Theorem 5.3** ([Oh2, Oh6]). *There exists a solution  $z : [0, 1] \rightarrow T^* N$  of  $\dot{z} = X(t, z)$  such that  $z(0) = q$ ,  $z(1) \in o_N$  and  $\mathcal{A}_H^{cl}(z) = \rho^{lag}(H; \{q\})$  whether or not  $\phi_H^1(o_N)$  intersects  $T_q^* N$  transversely.*

We summarize the main properties of  $f_H$  established in [Oh2].

**Theorem 5.4** ([Oh2]). *When the Hamiltonian  $H = H(t, x)$  such that  $L = \phi_H^1(o_N)$  is given, there is a canonical lift  $f_H$  defined by  $f_H(q) := \rho^{lag}(H; \{q\})$  that satisfies*

$$f_H \circ \pi(x) = h_H(x) = \mathcal{A}_H^{cl}(z_x^H) \quad (5.4)$$

*for some Hamiltonian chord  $z_x^H$  ending at  $x \in T_q^* N$ . This  $f_H$  satisfies the following property in addition*

$$\|f_H - f_K\|_\infty \leq \|H - K\|. \quad (5.5)$$

An immediate corollary of this theorem is

**Corollary 5.5.** *If  $H_i$  converges in  $L^{(1, \infty)}$ , then  $f_{H_i}$  converges uniformly.*

Based on this corollary, we will just denote the limit continuous function by

$$f_H := \lim_{i \rightarrow \infty} f_{H_i} \quad (5.6)$$

when  $H_i \rightarrow H$  in  $L^{(1,\infty)}$ -topology, and call it the basic phase function of the topological Hamiltonian  $H$  or of the  $C^0$ -Lagrangian submanifold  $L_H = \phi_H^1(o_N)$ .

Note that  $\pi_H = \pi|_{L_H} : L_H = \phi_H^1(o_N) \rightarrow N$  is surjective for all  $H$  (see [LS] for its proof) and so  $\pi_H^{-1}(\pi_H^{-1}(q)) \subset o_N$  is a non-empty compact subset of  $o_N \cong N$ . Therefore we can regard the ‘inverse’  $\pi_H^{-1} : N \rightarrow L_H \subset T^*N$  as a everywhere defined multivalued section of  $\pi : T^*N \rightarrow N$ .

We introduce the following general definition

**Definition 5.1.** Let  $L \subset T^*N$  be a Lagrangian submanifold projecting surjectively to  $N$ . We call a single valued section  $\sigma$  of  $T^*N$  with values lying in  $L$  a *Lagrangian selector* of  $L$ .

For any given Lagrangian selector  $\sigma$  of  $L = L_H = \phi_H^1(o_N)$ , we define the map  $\varphi^\sigma : N \rightarrow N$  to be

$$\varphi^\sigma(q) = (\phi_H^1)^{-1}(\sigma(q)).$$

Recall that the graph  $G_{f_H}$  is a subset of the front  $W_{R_H}$  of  $R_H$  and for a generic choice of  $H$  the set  $\text{Sing } f_H \subset N$  consists of the crossing points of the two different branches and the cusp points of the front of  $W_{R_H}$ . Therefore it is a set of measure zero in  $N$ . (See [El], [PPS], for example.) Once the graph selector  $f_H$  of  $L_H$  is picked out, it provides a natural Lagrangian selector defined by

$$\sigma_H(q) := \text{Choice}\{x \in L_H \mid \pi(x) = q, \mathcal{A}_H^{\text{cl}}(z_x^H) = f_H(q)\}$$

via the axiom of choice where Choice is a choice function. It satisfies

$$\sigma_H(q) = df_H(q) \quad (5.7)$$

whenever  $df_H(q)$  is defined. We call this particular Lagrangian selector of  $L_H$  the *basic Lagrangian selector* and the pair  $(\sigma_H, f_H)$  the *basic wave front* of the Lagrangian submanifold  $\phi_H^1(o_N)$ .

The general structure theorem of the wave front (see [El], [PPS] for example) proves that the section  $\sigma_H$  is a differentiable map on a set of full measure for a generic choice of  $H$  which is, however, *not necessarily continuous*: This is because as long as  $q \in N \setminus \text{Sing } f_H$ , we can choose a small open neighborhood of  $U \subset N \setminus \text{Sing } f_H$  of  $q$  and  $V \subset L_H = \phi_H^1(o_N)$  of  $x \in V$  with  $\pi(x) = q$  so that the projection  $\pi|_V : V \rightarrow U$  is a diffeomorphism.

Then we define the *mass transfer map*  $\varphi^H : N \rightarrow N$  by

$$\varphi^H(q) = (\phi_H^1)^{-1}(\sigma_H(q)). \quad (5.8)$$

The map  $\varphi^H$  is *measurable, but not necessarily continuous*, which is however differentiable on a set of full measure for a generic choice of  $H$ . And from its definition, it is surjective if and only if the Lagrangian submanifold  $\phi_H^1(o_N)$  is a graph of an exact one-form. On the other hand, the map  $\varphi^H$  may not be continuous along the subset  $\text{Sing } f_H \subset N$  which is a set of measure zero. By definition, we have

$$f_H(q) = \mathcal{A}_H^{\text{cl}}\left(z_H^{\varphi^H(q)}\right) = \tilde{h}_H(\varphi^H(q)). \quad (5.9)$$

This relationship between  $f_H$  and  $\tilde{h}_H$  is the reason why we introduce the transfer map  $\varphi^H$ .

The following lemma is obvious from the definition of  $\varphi^H$ . We note

$$d_H(\phi_H^1(o_N), o_N) \leq \text{osc}_{C^0}(\phi_H^1; o_N)$$

where  $d_H(\phi_H^1(o_N), o_N)$  is the Hausdorff distance.

**Lemma 5.6.** *We have*

$$d(\varphi^H(x), x) \leq d_H(\phi_H^1(o_N), o_N) + \text{osc}_{C^0}(\phi_H^1; o_N) \leq 2\text{osc}_{C^0}(\phi_H^1; o_N)$$

for all  $x \in N_0$ . In particular, if  $\text{osc}_{C^0}(\phi_H^1; o_N) \rightarrow 0$ , then  $\max_{x \in N_0} d(\varphi^H(x), x) \rightarrow 0$  uniformly over  $x \in N_0$ .

## 6. SINGULAR LOCUS OF THE BASIC PHASE FUNCTION AND CLIFF-WALL SURGERY

In this section, we consider general cotangent bundle  $T^*N$  of arbitrary closed manifold  $N$ . We first recall two important properties of the Liouville one-form  $\theta$  in this regard:

- (1)  $\theta$  identically vanishes on any conormal variety. (See [Oh2, KO1] for the explanation on the importance of this fact in relation to the Lagrangian Floer theory on the cotangent bundle.)
- (2) For any one form  $\alpha$  on  $N$ , we have  $\hat{\alpha}^*\theta = \alpha$  where  $\hat{\alpha} : N \rightarrow T^*N$  is the section map associated to the one-form  $\alpha$  as a section of  $T^*N$ . In particular, we have

$$\sigma_F^*\theta = df_F$$

on  $N \setminus \text{Sing}(\sigma_F)$  and on each stratum of  $\text{Sing}(\sigma_F)$ .

We note that the singular locus  $S(\sigma_F) \subset \Delta$  is a subset of the *bifurcation diagram* of the Lagrangian submanifold  $\phi_F^1(o_N)$ : The bifurcation diagram is the union of the caustic and the Maxwell set where the latter is the set of points of which merge the different branches of the generating function  $h$ . (See section 4 [G1] for the definition of bifurcation diagram of Lagrangian submanifold  $L \subset T^*N$  in general.)

For a generic  $F$ ,  $S(\sigma_F)$  is stratified into a finite union of smooth submanifolds

$$\bigcup_{k=1}^n S_k(\sigma_F), \quad S_k(\sigma_F) = \text{Sing}_k(\sigma_F), \quad n = \dim N$$

(see [A1, El, G1] e.g., for such a result) so that its conormal variety  $\nu^*S(\sigma_F)$  can be defined as a finite union of conormals of the corresponding strata. Each stratum  $\text{Sing}_k(\sigma_F)$  has codimension  $k$  in  $\Delta$ . The stratum for some  $k$  could be empty. (See [KS]. See also [Ka, KO2], [NZ, N] for the usages of such conormal varieties in relation to Lagrangian Floer theory.)

In  $\dim M = 2$ , there are two strata to consider, one  $S_1(\sigma_F)$  and the other  $S_2(\sigma_F)$ .

For  $k = 1$ , each given point  $q \in S_1(\sigma_F)$  has a neighborhood  $A(q) \subset N$  such that  $A(q) \setminus S_1(\sigma_F)$  has two components. We also note that  $\Sigma_F$  carries a natural orientation induced from  $N$  by projection when  $N$  is orientable and so defines an integral current in the sense of geometric measure theory [Fe]. When  $N$  is oriented,  $S_1(F)$  is also orientable as a finite union of smooth hypersurface. We fix any orientation on  $S_1(F)$ .

We denote by  $A^\pm(q)$  the closure of each component of  $A(q) \setminus S_1(\sigma_F)$  in  $A(q)$  respectively. Here we denote by  $A^+(q)$  the component whose boundary orientation

on  $\partial A^+(q)$  coincides with that of the given orientation on  $S_1(F)$  and by  $\partial A^-(q)$  the other one. Then each of  $A^\pm(q)$  is an open-closed domain with the same boundary

$$\partial A^\pm(q) = A(q) \cap S_1(\sigma_F).$$

Denote

$$df_F^\pm(q) = \lim_{p_\pm \rightarrow q} df_F(p_\pm) \quad (6.1)$$

obtained by taking the limit on  $A^\pm(q)$  respectively. The limits are well-defined from the definition of  $\sigma_F$  since  $\text{Im } \sigma_F = \text{Im } \widehat{df_F} \subset \phi_F^1(o_N)$  where  $\phi_F^1(o_N)$  is a smooth closed submanifold in  $T^*N$ .

We now prove the following theorem. We refer to [G1], [ZR] for a related statement.

**Theorem 6.1.** *Let  $q \in S_1(F)$ . Then*

$$df_F^-(q) - df_F^+(q) \in T_q^*N,$$

*which is contained in the conormal space  $\nu_q^*[S_1(\sigma_F); N] \subset T_q^*N$ .*

*Proof.* Let  $\vec{v} \in T_q S_1(\sigma_F)$  be any given tangent vector. Choose a smooth curve  $\gamma : (-\varepsilon, \varepsilon) \rightarrow S_1(\sigma_F)$  with  $\gamma(0) = q$ . For any given sufficiently small  $\delta \geq 0$ , we define a family of  $\delta$ -shifted curves

$$\gamma_\delta^\pm(t) = \exp_{\gamma(t)}(\pm \delta \vec{n}(t)),$$

where  $\exp$  is the normal exponential map of  $S_1(\sigma_F)$  in  $N$  and  $\vec{n}(t)$  is the unit normal vector thereof at  $\gamma(t)$  towards the domain  $A^+(q)$ . Then  $\gamma_\delta^+$  is mapped into  $\text{Int } A^+(q)$  and  $\gamma_\delta^-$  into  $\text{Int } A^-(q)$  for all sufficiently small  $\delta > 0$ . Note

$$\gamma_0^\pm(t) = \gamma(t)$$

for  $\delta = 0$ . Since  $f_F : N \rightarrow \mathbb{R}$  is a continuous function, we have the uniform convergence

$$f_F(\gamma_\delta^+(t)) - f_F(\gamma_\delta^-(t)) \rightarrow 0$$

as  $\delta \rightarrow 0$  over  $t \in (-\varepsilon, \varepsilon)$ . Furthermore since  $f_F$  is smooth up to the boundary on each of  $A^\pm(q)$  and  $df_F$  is uniformly differentiable up to the boundary of  $A^\pm(q)$  for either of  $\pm$ ,

$$\begin{aligned} f_F(\gamma_\delta^\pm(t)) &= f_F(\gamma_\delta^\pm(0)) + t df_F(\gamma_\delta^\pm(0))((\gamma_\delta^\pm)'(0)) + O(|t|^2) \\ &= f_F(\gamma_\delta^\pm(0)) + t df_F(\gamma_\delta^\pm(0)) \circ D \exp_{\gamma(0)}(\pm \delta \vec{n}(0))(\gamma'(0)) + O(|t|^2) \end{aligned}$$

where  $|O(|t|^2)| \leq C|t|^2$  for a constant  $C > 0$  uniformly over  $\delta \geq 0$  and  $t \in (-\varepsilon, \varepsilon)$ . Here  $D \exp_p(\vec{n})(\vec{v})$  is the derivative

$$D \exp_p(\vec{n})(\vec{v}) := \left. \frac{d}{dt} \right|_{t=0} \exp_{\gamma(t)}(\vec{n}), \quad \vec{v} = \gamma'(0), \quad \gamma(0) = p,$$

which is nothing but the covariant derivative of the Jacobi field along the geodesic  $t \mapsto \exp_p(tv)$  with the initial vector  $\vec{n}$  at  $p$ . (See [K] for an elegant exposition on the detailed study of exponential maps.) By letting  $\delta \rightarrow 0$  and using the uniformity of the constant  $C$  and the continuity of  $f_F$ , we obtain

$$\begin{aligned} f_F(\gamma(t)) &= f_F(q) + \lim_{\delta \rightarrow 0} (t df_F(\gamma_\delta^\pm(0))((\gamma_\delta^\pm)'(0))) + O(|t|^2) \\ &= f_F(q) + t \lim_{\delta \rightarrow 0} df_F^\pm(\gamma_\delta^\pm(0)) \left( D \exp_{\gamma(0)}(\pm \delta \vec{n}(0))((\gamma_\delta^\pm)'(0)) \right) + O(|t|^2). \end{aligned}$$

Then by taking the difference of two equations for  $\pm$  and dividing by  $t$ , utilizing the convergence  $(\gamma_\delta^\pm)'(0) \rightarrow \gamma'(0)$  as  $\delta \rightarrow 0$  and then evaluating at  $t = 0$ , we obtain

$$0 = \lim_{\delta \rightarrow 0} \left( df_F^+(\gamma_\delta^+(0)) \circ D \exp_{\gamma(0)}(+\delta \vec{n}(0)) - df_F^-(\gamma_\delta^-(0)) \circ D \exp_{\gamma(0)}(-\delta \vec{n}(0)) \right) (\gamma'(0)).$$

Recall that  $\gamma(0) = p$  and  $\gamma_\delta^\pm(0) \rightarrow p$ , and  $D \exp_p(\pm \delta \vec{n}(0))$  converges to  $D \exp_p(\vec{0})$  as  $\delta \rightarrow 0$ , which is nothing but the identity map on  $\nu_q S_1(\sigma_F)$  by the standard fact on the exponential map (see [K]). Therefore from this last equality, we derive

$$(df_F^+(q) - df_F^-(q))(\vec{v}) = 0$$

by the definition of  $df_F^\pm(q)$ . Since this holds for all  $\vec{v} \in T_q S_1(\sigma_F)$ , the proposition for  $k = 1$  is proved.  $\square$

The boundary orientations of the two components arising from that of  $\Sigma_F$ , which in turn is induced from that of  $N$  via  $\pi_1$  have opposite orientations. We call the one whose projection to  $S_1(\sigma_F)$  under  $\pi_1$  coinciding with the given orientation the *upper branch* and the one with the opposite one the *lower branch* and denote them by

$$\partial^+ \Sigma_F, \partial^- \Sigma_F$$

respectively.

Now let  $L_q$  be the line segment connecting the two vectors  $df_F^\pm(q)$ , i.e.,

$$L_q : u \in [0, 1] \mapsto df_F^+(q) + u(df_F^-(q) - df_F^+(q)) \subset T_q^* N. \quad (6.2)$$

This is an affine line that is parallel to the conormal space  $\nu_q^* S_1(\sigma_F)$ . Therefore the union

$$\Sigma_{F;[-+]} := \bigcup_{q \in S_1(\sigma_F)} L_q \quad (6.3)$$

is contained in the translated conormal

$$df_F^+ + \nu^*[S_1(\sigma_F); N] \quad (6.4)$$

Here the bracket  $[-+]$  stands for the line segment  $L_q$ , and  $\nu^*[S_1(\sigma_F); N]$  is the conormal bundle of  $S_1(\sigma_F)$  in  $N$ . We would like to point out that since  $df_F^+(q) - df_F^-(q) \in \nu^*[S_1(\sigma_F); N]$  we have the equality

$$df_F^+(q) + \nu_q^*[S_1(\sigma_F); N] = df_F^-(q) + \nu_q^*[S_1(\sigma_F); N]$$

for all  $q \in S_1(\sigma_F)$ . Therefore we can simply write (6.4) as

$$df_F + \nu^*[S_1(\sigma_F); N] \quad (6.5)$$

unambiguously.

**Definition 6.1** (Basic Lagrangian selector chain). We denote by  $\sigma_F$  the chain whose support is given by

$$\text{supp}(\sigma_F) := \overline{\Sigma}_F \quad (6.6)$$

with the orientation given as above, and define its *micro-support* by

$$SS(\sigma_F) := \overline{df_F + \nu^*[S_1(\Sigma_F); N]} \quad (6.7)$$

imitating the notation from [KS].

The two components of  $\partial\sigma_F$  associated to each connected component of  $S_1(\sigma_F)$  are the graphs of  $df_F^\pm$  for the functions  $f_F^\pm$  near  $S_1(\sigma_F)$ .

Note that each connected component of  $S_1(\sigma_F)$  gives rise to two components of  $\partial\sigma_{F;[-+]}\cap\sigma_F$ . We can bridge the ‘cliff’ between the two branches of  $\partial\sigma_F$  over each connected component of  $S_1(\sigma_F)$  and

**Definition 6.2** (Cliff wall chain). We define a ‘cliff wall’ chain  $\sigma_{F;[-+]}$  whose support is given by the union

$$\Sigma_{F;[-+]} = \bigcup_{q \in S_1(\sigma_F)} L_q$$

Then we define the chain  $\sigma_{F;[-+]}$  similarly as we define  $\sigma_F$  by taking its closure in  $T^*N$ .

We emphasize that  $\sigma_{F;[-+]}$  lies outside the Lagrangian submanifold  $\phi_F^1(o_N)$ .

By definition, its tangent space at  $x = (q, u)$  has natural identification with

$$T_x \Sigma_{F;[-+]} \cong \nu_q^* S_1(\sigma_F) \oplus T_q S_1(\sigma_F).$$

Due to Theorem 6.1, it carries a natural direct sum orientation

$$o_{\Sigma_{F;[-+]}}(q) = \{df_F^-(q) - df_F^+(q)\} \oplus o_{S_1(\sigma_F)}(q).$$

Therefore  $\Sigma_{F;[-+]}$  carries a natural orientation and defines a current. Under the natural identification of  $T_q N$  with  $T_q^* N$  by the dual pairing, which induces an identification

$$\nu_q^* S_1(\sigma_F) \oplus T_q S_1(\sigma_F) \cong \nu_q S_1(\sigma_F) \oplus T_q S_1(\sigma_F)$$

as an oriented vector space. Then we have the relation

$$\partial \Sigma_F = -\partial \Sigma_{F;[-+]}. \quad (6.8)$$

**Remark 6.3.** (1) We would like to note that the singular locus  $S(\sigma_F) \subset \Delta$  is a subset of the *bifurcation diagram* of the Lagrangian submanifold  $\phi_F^1(o_N)$ : The bifurcation diagram is the union of the caustic and the Maxwell set where the latter is the set of points of which merge the different branches of the generating function  $h$ . (See section 4 [G1] for the definition of bifurcation diagram of Lagrangian submanifold  $L \subset T^*N$  in general.) But this detailed structure does not play any role in our proof except the one described.

(2) However we would like to note that each fiber of  $SS(\sigma_F)$  is an affine space

$$df_F(q) + \nu_q^*[S_1(\Sigma_F); N]$$

at  $q \in S_1(\Sigma_F)$ , not a linear space. In fact, if we incorporate the orientation into consideration, one can refine this definition further to the ‘half space’ instead of the full affine space. We denote this refinement by  $SS^+(\sigma_F)$ . Then at a point  $q$  in the lower dimensional strata, it will be a ‘wedge domain’, i.e., the intersection of several space of this type. (See [KO1, KO2] for a usage of such domains in their quantization program of Eilenberg-Steenrod axiom.) We will come back to further discussion on the detailed structure of singularities elsewhere.

Next we consider the case of  $S_2(\sigma_F)$  and its relationship with  $\sigma_F$  and  $S_1(\sigma_F)$ . Note that for a generic choice of  $F$ ,  $S_2(\sigma_F)$  consists of a finite number of points in

$N$  consisting of either a caustic point or a triple intersection point of the Maxwell set (see [A1], section 4 [G1] and 7.1 [ZR]).

The following proposition can be also derived from the general structure theorem of generic singularities of Lagrangian maps. We restrict the proposition to  $\dim M = 2$  here postponing the precise statement for the high dimensional cases elsewhere.

**Proposition 6.2.** *Assume  $n = 2$ . For a generic choice of  $F$ , the boundary of  $\sigma_F + \sigma_{F;[-+]}$  is a finite union of triangles each of which is formed by the three line segments  $L_q$  given in (6.2) associated to a triple intersection point  $q$  of  $S(\sigma_F)$  contained in  $S_2(\sigma_F)$ . Furthermore each triangle is the boundary of a 2-simplex contained in the fiber  $T_q^*N$ .*

*Proof.* This is an immediate consequence of the classification theorem of generic singularities in dimension 2 of Lagrangian maps originally proved by Arnold [A1]. (See also p. 55 and Figure 43 [A3], section 4 [G1] and 7.1 [ZR].)  $\square$

Now we define  $\sigma_{F;\Delta^2}$  to be the union of these 2 simplices, and set

$$\sigma_F^{add} = \sigma_F + \sigma_{F;[-+]} + \sigma_{F;\Delta^2}.$$

Then by construction,  $\sigma_F^{add}$  forms a mod-2 cycle.

This finishes the description of the basic Lagrangian cycle. A similar description can be given in the higher dimensional cases, which we will study elsewhere. This enables us to define the following important Lagrangian cycle which will play a crucial role in our homological integration in the next section.

**Definition 6.4** (Basic Lagrangian cycle and cliff-wall surgery). Let  $\dim M = 2$ . We call the cycle  $\sigma_F^{add}$  the *basic Lagrangian cycle* of  $\phi_F^1(o_N)$  (associated to the basic Lagrangian selector  $\sigma_F$ ). We call the replacement of  $\phi_F^1(o_N)$  by the  $\Sigma_F^{add}$  the *cliff-wall surgery* of the  $\phi_F^1(o_N)$ .

**Remark 6.5.** (1) We also refer to [KO1, Ka, KO2] for a usage of the general conormal variety of an open-closed domain with boundary and corners, which also naturally occurs in micro-local analysis and in stratified Morse theory [KS].  
 (2) The basic Lagrangian cycle seems to be a good replacement of non-graph type Lagrangian submanifold  $\phi_F^1(o_N)$  in general for the study of various questions arising in Hamiltonian dynamics and symplectic topology. We hope to elaborate this point elsewhere.

Note that when  $\bar{d}(\phi_F^1, id)$  is sufficiently small, each simplex  $\Delta_{\mathbf{q}}^2$  is contained in the product  $U_y \times U_y \subset V_{\Delta} \subset T^*\Delta$  associated to a Darboux neighborhood  $U_y \subset M$  in the given Darboux family  $\Phi$ .

## 7. LAGRANGIAN FLOER HOMOLOGY AND SPECTRAL INVARIANTS

In this section, we first briefly recall the construction of Lagrangian spectral invariants  $\rho^{lag}(H; a)$  for  $L_H = \phi_H^1(o_N)$  performed by the author in [Oh3]. A priori, this invariant may depend on  $H$ , not just on  $L_H$  itself. In [Oh3], we prove that

$$\rho^{lag}(H; a) = \rho^{lag}(F; a) \quad (7.1)$$

for all  $a \in H^*(N; \mathbb{Z})$  if  $L_H = L_F$ , but modulo the addition of a constant and then somewhat ad-hoc normalization to remove this ambiguity of a constant.



**7.1. Definition of Lagrangian spectral invariants.** Consider the zero section  $o_N$  and the space

$$\mathcal{P}(o_N, o_N) = \{\gamma : [0, 1] \rightarrow T^*N \mid \gamma(0), \gamma(1) \in o_N\}.$$

The set of generators of  $CF(H; o_N, o_N)$  is that of solutions

$$\dot{z} = X_H(t, z(t)), \quad z(0), z(1) \in o_N$$

and its Floer differential is defined by counting the number of solutions of

$$\begin{cases} \frac{\partial u}{\partial \tau} + J \left( \frac{\partial u}{\partial t} - X_H(u) \right) = 0 \\ u(\tau, 0), u(\tau, 1) \in o_N. \end{cases} \quad (7.2)$$

An element  $\alpha \in CF(H; o_N, o_N)$  is expressed as a finite sum

$$\alpha = \sum_{z \in \text{Chord}(H; o_N, o_N)} a_z [z], \quad a_z \in \mathbb{Z}.$$

We denote the *level* of the chain  $\alpha$  by

$$\lambda_H(\alpha) := \max_{z \in \text{supp } \alpha} \{\mathcal{A}_H^{\text{cl}}(z)\}. \quad (7.3)$$

For given non-zero cohomology class  $a \in H^*(N, \mathbb{Z})$ , we consider its Poincaré dual  $[a]^b := PD(a) \in H_*(N, \mathbb{Z})$  and its image under the canonical isomorphism

$$\Phi : H_*(N, \mathbb{Z}) \rightarrow HF_*(H, J; o_N, o_N).$$

**Definition 7.1.** Let  $(H, J)$  be a Floer regular pair relative to  $(o_N, o_N)$  and let  $(CF(H), \partial_{(H, J)})$  be its associated Floer complex. For any  $0 \neq a \in H^*(N, \mathbb{Z})$ , we define

$$\rho^{\text{lag}}(H; a) = \inf_{\alpha \in \Phi([a]^b)} \{\lambda_H(\alpha)\}. \quad (7.4)$$

One important result is the following basic property, called *spectrality* in [Oh6], which is not explicitly stated in [Oh2] but can be easily derived by a compactness argument. (See the proof in [Oh6] given in the Hamiltonian context.)

**Proposition 7.1.** *Let  $H = H(t, x)$  be any, not necessarily nondegenerate, smooth Hamiltonian. Then for any  $0 \neq a \in H^*(N, \mathbb{Z})$ , there exists a point  $x \in L_H \cap o_N$  such that*

$$\mathcal{A}_H^{\text{cl}}(z_x^H) = \rho^{\text{lag}}(H; a).$$

*In particular,  $\rho^{\text{lag}}(H; a) \in \text{Spec}(H; N)$ .*

**7.2. Comparison of two Cauchy-Riemann equations.** So far we have looked at the Hamiltonian-perturbed Cauchy-Riemann equation (7.2), which we call the *dynamical version* as in [Oh2].

On the other hand, one can also consider the *genuine* Cauchy-Riemann equation

$$\begin{cases} \frac{\partial v}{\partial \tau} + J^H \frac{\partial v}{\partial t} = 0 \\ v(\tau, 0) \in \phi_H^1(o_N), v(\tau, 1) \in o_N \end{cases} \quad (7.5)$$

for the path  $u : \mathbb{R} \rightarrow \mathcal{P}(o_N, L)$  where  $L = \phi_H^1(o_N)$  and

$$\mathcal{P}(o_N, L) = \{\gamma : [0, 1] \rightarrow T^*N \mid \gamma(0) \in L, \gamma(1) \in o_N\}$$

and  $J_t^H = (\phi_H^t \phi_H^{-1})_* J_t$ . We call this version the *geometric version*.

We now describe the geometric version of the Floer homology in some more details. We refer readers to [Oh2] for the discussion on the further comparison

of the two versions in the point of moduli spaces and others. The upshot is that there is a filtration preserving isomorphisms between the dynamical version and the geometric version of the Lagrangian Floer theories.

We denote by  $\widetilde{\mathcal{M}}(L_H, o_N; J^H)$  the set of finite energy solutions and  $\mathcal{M}(L_H, o_N; J^H)$  to be its quotient by  $\mathbb{R}$ -translations. This gives rise to the geometric version of the Floer homology  $HF_*(o_N, \phi_H(o_N), \widetilde{J})$  of the type [Fl1, Oh3] whose generators are the intersection points of  $o_N \cap \phi_H(o_N)$ . An advantage of this version is that it depends only on the Lagrangian submanifold  $L = \phi_H(o_N)$ , only loosely on  $H$ . (The author proved in [Oh3] that  $\rho(H; a)$  is the invariant of  $L_H = \phi_H(o_N)$  up to this normalization by comparing these two versions of the Floer theory in [Oh2, Oh3].)

The following is a straightforward to check but is a crucial lemma.

**Lemma 7.2.** *Let  $L = \phi_H^1(o_S)$ .*

- (1) *The map  $\Phi_H : o_N \cap L \rightarrow \text{Chord}(H; o_N, o_N)$  defined by*

$$x \mapsto z_x^H(t) = \phi_H^t((\phi_H^1)^{-1}(x))$$

*gives rise to the one-one correspondence between the set  $o_N \cap L \subset \mathcal{P}(o_N, L)$  as constant paths and the set of solutions of Hamilton's equation of  $H$ .*

- (2) *The map  $a \mapsto \Phi_H(a)$  also defines a one-one correspondence from the set of solutions of (7.2) and that of*

$$\begin{cases} \frac{\partial v}{\partial \tau} + J^H \frac{\partial v}{\partial t} = 0 \\ v(\tau, 0) \in \phi_H(o_N), v(\tau, 1) \in o_N \end{cases} \quad (7.6)$$

*where  $J^H = \{J_t^H\}$ ,  $J_t^H := (\phi_H^t(\phi_H^1)^{-1})^* J_t$ . Furthermore, (7.6) is regular if and only if (7.2) is regular.*

Once we have transformed (7.2) to (7.6), we can further deform  $J^H$  to the constant family  $J_0$  and consider

$$\begin{cases} \frac{\partial v}{\partial \tau} + J_0 \frac{\partial v}{\partial t} = 0 \\ v(\tau, 0) \in \phi_H(o_N), v(\tau, 1) \in o_N. \end{cases} \quad (7.7)$$

This latter deformation preserves the filtration of the associated Floer complexes [Oh2]. A big advantage of considering this equation is that it enables us to study the behavior of spectral invariants for a sequence of  $L_i$  converging to  $o_N$  in Hamiltonian distance.

The following proposition provides the action functional associated to the equation (7.6), (7.7), which will give a natural filtration associated Floer homology  $HF(L, o_N)$ .

**Proposition 7.3.** *Let  $L$  and  $h_L$  be as in Lemma 2.1. Let  $\Omega(L, o_N; T^*N)$  be the space of paths  $\gamma : [0, 1] \rightarrow \mathbb{R}$  satisfying  $\gamma(0) \in L, o_N, \gamma(1) \in o_N$ . Consider the effective action functional*

$$\mathcal{A}^{\text{eff}}(\gamma) = \int \gamma^* \theta + h_H(\gamma(0)).$$

*Then  $d\mathcal{A}^{\text{eff}}(\gamma)(\xi) = \int_0^1 \omega(\xi(t), \dot{\gamma}(t)) dt$ . In particular,*

$$\mathcal{A}^{\text{eff}}(c_x) = h_H(x) = \mathcal{A}_H^{\text{cl}}(z_x^H) \quad (7.8)$$

*for the constant path  $c_x \equiv x \in L \cap o_N$  i.e., for any critical path  $c_x$  of  $\mathcal{A}^{\text{eff}}$ .*

We would like to highlight the presence of the ‘boundary contribution’  $h_H(\gamma(0))$  in the definition of the effective action functional above: This addition is needed to make the Cauchy-Riemann equation (7.5) or (7.7) into a *gradient trajectory equation* of the relevant action functional. We refer readers to section 2.4 [Oh2] and Definition 3.1 [KO1] and the discussion around it for the upshot of considering the effective action functional and its role in the study of Cauchy-Riemann equation.

**7.3. Triangle inequality for Lagrangian spectral invariants.** We recall from, [Sc], [Oh6] that the triangle inequality of the Hamiltonian spectral invariants

$$\rho^{ham}(H \# F; a \cdot b) \leq \rho^{ham}(H; a) + \rho^{ham}(F; b)$$

for the product Hamiltonian  $H \# F$  relies on the homotopy invariance property of spectral invariants which in turn relies on the existence of canonical normalization procedure of Hamiltonians on closed  $(M, \omega)$  which is nothing but the *mean normalization*. On the other hand, one can directly prove

$$\rho^{ham}(H * F; a \cdot b) \leq \rho^{ham}(H; a) + \rho^{ham}(F; b)$$

more easily for the concatenated Hamiltonian. (See e.g., [FOOO3] for the proof.) Once we have the latter inequality, we can derive the former from the latter again by the homotopy invariance property of  $\rho^{ham}(\cdot; a)$  for the *mean-normalized Hamiltonians*.

When one attempts to assign an invariant of Lagrangian submanifold  $\phi_H^1(o_N)$  itself out of the spectral invariant  $\rho^{lag}(H; a)$ , one has to choose a normalization of the Hamiltonian *relative to* the Lagrangian submanifold. Since there is no canonical normalization unlike the Hamiltonian case, the invariance property of Lagrangian spectral invariants and so the triangle inequality is somewhat more nontrivial than the case of Hamiltonian spectral invariants. In this subsection, we clarify these issues of invariance property and of the triangle inequality.

The following parametrization independence follows immediately from the construction of Lagrangian spectral invariants and  $L^{(1, \infty)}$ -continuity of  $H \mapsto \rho^{lag}(H; a)$ .

**Lemma 7.4.** *Let  $H = H(t, x)$  be any, not necessarily nondegenerate, smooth Hamiltonian and let  $\chi : [0, 1] \rightarrow [0, 1]$  a reparameterization function with  $\chi(0) = 0$  and  $\chi(1) = 1$ . Then*

$$\rho^{lag}(H; a) = \rho^{lag}(H^\chi; a)$$

where  $H^\chi(t, x) = \chi'(t)H(\chi(t), x)$ .

We first recall the following triangle inequality which was essentially proved in [Oh3]. (See Theorem 6.4 and Lemma 6.5 [Oh3]. In [Oh3], the cohomological version of the Floer complex was considered and hence the opposite inequality is stated. Other than this, the same proof can be applied here.)

**Proposition 7.5.** *Let  $H, F \in \mathcal{PC}_{asc}^\infty(T^*N; \mathbb{R})$ , and assume  $F$  is autonomous. Then we have*

$$\rho^{lag}(H \# F; ab) \leq \rho^{lag}(H; a) + \rho^{lag}(F; b). \quad (7.9)$$

Monzner, Vichery, and Zapolsky [MVZ] proved the following form of the triangle inequality which uses the concatenated Hamiltonian  $H * F$  instead of the product Hamiltonian  $H \# F$ .

**Proposition 7.6** (Proposition 2.4 [MVZ]). *Suppose  $H(1, x) \equiv F(0, x)$  and  $H * F$  be the concatenated Hamiltonian. Then*

$$\rho^{lag}(H * F; ab) \leq \rho^{lag}(H; a) + \rho^{lag}(F; b) \quad (7.10)$$

for all  $a, b \in H^*(N)$ .

In particular, this proposition applies to all pairs  $H, F$  which are boundary flat.

**Remark 7.2.** We suspect that (7.9) holds even for the non-autonomous  $F$  as in the Hamiltonian case but we did not check this, since it is not needed in the present paper.

**7.4. Assigning spectral invariants to Lagrangian submanifolds.** In this subsection, we identify a class, denoted by  $\mathcal{PC}_{(B;e)}^\infty$ , of Hamiltonians  $H$  among those satisfying  $\phi_H^1(o_N) = \phi_F^1(o_N)$ , such that the equality

$$\rho^{lag}(H; a) = \rho^{lag}(F; a)$$

holds for all  $H, F \in \mathcal{PC}_{(B;e)}^\infty$ . As the notation suggests, the class depends on the subset  $B \subset N$  and the real number  $e \in \mathbb{R}$ .

We start with the following proposition. The proof closely follows that of Lemma 2.6 [MVZ] which uses Proposition 7.6 in a significant way. We need to modify their proof to obtain a somewhat stronger statement, which replaces the condition “ $\phi_H^1 = \phi_F^1$ ” used in [MVZ] by the conditions put in this proposition.

**Proposition 7.7** (Compare with Lemma 2.6 [MVZ]). *Let  $H, F \in \mathcal{PC}_{asc}^\infty(T^*N; \mathbb{R})$  be boundary-flat. Suppose in addition  $H, F$  satisfy the following:*

- (1)  $\phi_H^1(o_N) = \phi_F^1(o_N)$ ,
- (2)  $H \equiv c(t)$ ,  $F \equiv d(t)$  on a tubular neighborhood  $T \supset B$  in  $T^*N$  of a closed ball  $B \subset o_N$  where  $c(t), d(t)$  are independent of  $x \in T$ , and
- (3) they satisfy

$$\int_0^1 c(t) dt = \int_0^1 d(t) dt.$$

Then  $\rho^{lag}(H; a) = \rho^{lag}(F; a)$  holds for all  $a \in H^*(N, \mathbb{Z})$  without ambiguity of constant.

*Proof.* We consider the Hamiltonian path  $\phi_G : t \mapsto \phi_G^t$  with  $G = \tilde{F} * H$  with  $\tilde{F}(t, x) = -F(1-t, x)$ . This defines a loop of Lagrangian submanifold

$$t \mapsto \phi_G^t(o_N), \quad \phi_G^1(o_N) = o_N$$

and satisfies  $\phi_G^t|_B \equiv id$  and

$$G(t, q) = \begin{cases} -c(1-2t) & 0 \leq t \leq 1/2 \\ d(2t-1) & 1/2 \leq t \leq 1 \end{cases}$$

for all  $q \in B \subset T$  by definition  $G = \tilde{F} * H$ .

We claim  $\rho^{lag}(G; a) = 0$  for all  $0 \neq a \in H^*(N)$ . This will be an immediate consequence of the following lemma and the spectrality of numbers  $\rho^{lag}(G; a)$ .

**Lemma 7.8.** *The value  $\mathcal{A}_G^{cl}(z)$  does not depend on the Hamiltonian chord  $z \in \text{Chord}(G; o_N, o_N)$ . In particular,  $\mathcal{A}_G^{cl}(z) = 0$ .*

*Proof.* Recall that any Hamiltonian chord in  $\mathcal{Chord}(G; o_N, o_N)$  has the form

$$z(t) = z_G^q(t)$$

for some  $q \in o_N$ . Here we use the hypothesis  $\phi_G^1(o_N) = o_N$ . Consider any smooth path  $\alpha : [0, 1] \rightarrow o_N$  with  $\alpha(0) = q$ ,  $\alpha(1) = q'$ . Then

$$\mathcal{A}_G^{cl}(z_G^{q'}) - \mathcal{A}_G^{cl}(z_G^q) = \int_0^1 \frac{d}{du} \mathcal{A}_G^{cl}(z_G^{\alpha(u)}) du.$$

But a straightforward computation using the first variation formula (4.4) implies

$$\frac{d}{du} \mathcal{A}_G^{cl}(z_G^{\alpha(u)}) = \left\langle \theta, \frac{\partial}{\partial u}(\phi_G(\alpha(u))) \right\rangle - \left\langle \theta, \frac{\partial}{\partial u}(\alpha(u)) \right\rangle = 0 - 0 = 0$$

since  $\phi_G(\alpha(u))$ ,  $\alpha(u) \in o_N$ .

For the second statement, we have only to consider the constant path  $z \equiv c_q \in B$  for which

$$\begin{aligned} \mathcal{A}_G^{cl}(c_q) &= - \int_0^1 G(t, q) dt = \int_0^{1/2} c(1-2t) dt - \int_{1/2}^1 d(2t-1) dt \\ &= \int_0^1 c(t) dt - \int_0^1 d(t) dt = 0. \end{aligned}$$

This proves the lemma.  $\square$

Once we have the lemma, we can apply the triangle inequality (7.10)

$$\rho^{lag}(H; a) \leq \rho^{lag}(F; a) + \rho^{lag}(G; 1) = \rho^{lag}(F; a)$$

for any given  $a \in H^*(N)$ . By changing the role of  $H$  and  $F$  in the proof of the above lemma, we also obtain  $\rho^{lag}(\tilde{G}; 1) = 0$  and then obtain  $\rho^{lag}(F; a) \leq \rho^{lag}(H; a)$  by triangle inequality. This finishes the proof of the proposition.  $\square$

This proposition motivates us to introduce the following definitions

**Definition 7.3.** For each given  $B \subset N$ , we define

$$\mathfrak{Iso}_B(o_N; T^*N) = \{L \in \mathfrak{Iso}(o_N; T^*N) \mid o_N \cap L \supset B\}.$$

When a function  $c : [0, 1] \rightarrow \mathbb{R}$  is given in addition, we define

$$\begin{aligned} \mathcal{PC}_{(B;e)}^\infty &= \{H \in \mathcal{PC}_{asc}^\infty \mid H_t \equiv c(t) \text{ on a neighborhood of } B \text{ in } T^*N \\ &\quad \text{and } \int_0^1 c(t) dt = e\}. \end{aligned}$$

With these definitions, the proposition enables us to unambiguously define the following spectral invariant attached to  $L$ .

**Definition 7.4.** Suppose  $L \in \mathfrak{Iso}_B(o_N; T^*N)$  and let  $e \in \mathbb{R}$  be given. For each given such  $e$ , we define a spectral invariant of  $L \in \mathfrak{Iso}_B(o_N; T^*N)$  by

$$\rho^{(B;e)}(L; a) := \rho^{lag}(H; a), \quad L = \phi_H^1(o_N)$$

for a (and so any)  $H \in \mathcal{PC}_{(B;e)}^\infty$ .

With this definition, we have the following obvious lemma

**Lemma 7.9.** Let  $H \in \mathcal{PC}_{(B;e)}^\infty$ , then  $\tilde{H}, \overline{H} \in \mathcal{PC}_{(B;-e)}^\infty$ .

Then we prove the following duality statement of  $\rho^{(B;e)}$ .

**Proposition 7.10.** *Let  $H \in \mathcal{PC}_{(B;e)}^\infty$  and  $L = \phi_H^1(o_N)$ . We denote  $\tilde{L} = \phi_{\tilde{H}}^1(o_N) = \phi_{\tilde{H}}^1(o_N)$ . Then*

$$\rho^{(B;-e)}(\tilde{L}; 1) = -\rho^{(B;e)}(L; [pt]^\#). \quad (7.11)$$

*Proof.* By the above lemma,  $\tilde{H} \in \mathcal{PC}_{(B;-e)}^\infty$  and so  $\rho^{(B;-e)}(\tilde{L}; 1)$  is given by

$$\rho^{(B;-e)}(\tilde{L}; 1) = \rho^{lag}(\tilde{H}; 1)$$

by definition. But it was proven in [V1, Oh2, Oh3] that

$$\rho^{lag}(\tilde{H}; 1) = -\rho^{lag}(H; [pt]^\#) \quad (7.12)$$

which follows from the Poincaré duality argument, by studying the time-reversal flow of the Floer equation (1.8)  $\tilde{u}$  defined by  $\tilde{u}(\tau, t) = u(-\tau, 1 - t)$ . The map  $\tilde{u}$  satisfies the equation

$$\begin{cases} \frac{\partial \tilde{u}}{\partial \tau} + \tilde{J} \left( \frac{\partial \tilde{u}}{\partial t} - X_{\tilde{H}}(\tilde{u}) \right) = 0 \\ \tilde{u}(\tau, 0), \tilde{u}(\tau, 1) \in o_N. \end{cases}$$

Furthermore this equation is compatible with the involution of the path space

$$\iota : \Omega(o_N, o_N) \rightarrow \Omega(o_N, o_N)$$

defined by  $\iota(\gamma)(t) = \tilde{\gamma}(t)$  with  $\tilde{\gamma}(t) = \gamma(1 - t)$  and the action functional identity

$$\mathcal{A}_{\tilde{H}}^{cl}(\tilde{\gamma}) = -\mathcal{A}_H^{cl}(\gamma).$$

We refer to [Oh3] for the details of the duality argument in the Floer theory used in the derivation of (7.12).

On the other hand, by definition,

$$\rho^{lag}(H; [pt]^\#) = \rho^{(B;e)}(L; [pt]^\#)$$

since  $H \in \mathcal{PC}_{(B;e)}^\infty$ . This finishes the proof.  $\square$

## 8. COMPARISON THEOREM OF $f_H$ AND $\rho^{lag}(H; 1)$

We first remark that both  $\rho^{lag}(H; 1)$  and  $f_H$  remain unchanged under the change of  $H$  outside a neighborhood of  $\bigcup_{t \in [0,1]} \phi_H^t(o_N)$ .

The main theorem we prove in this section is the following whose proof occupies the entirety of this section.

**Theorem 8.1.** *For any Hamiltonian  $H \in \mathcal{PC}_{asc}^\infty$ ,*

$$\max f_H \leq \rho^{lag}(H; 1).$$

**Remark 8.1.** One might recall the general inequality  $\rho^{lag}(H; [pt]^\#) \leq \rho^{lag}(H; 1)$  and so wonder whether the inequality  $\rho^{lag}(H; [pt]^\#) \leq \min f_H$  from below would also hold. However this inequality fails to hold in general. See Example 9.4 [Oh2] which studies an example of Lagrangians on  $T^*S^1$ . In that example, one can check that  $\rho^{lag}(H; [pt]^\#) = 0$  which is realized by the level of the Floer cycle  $z_1 + z_3$  for the intersections  $z_1, z_3$  in the example. But the minimum of  $f_H$  is realized by a negative number at a non-smooth point of the function  $f_H$ .

We first recall the definition of the triangle product described in [Oh3], [FO] and put it into a more modern context in the general Lagrangian Floer theory such as in [FOOO1] and in other more recent literatures.

Let  $q \in N$  be given. Consider the Hamiltonians  $H : [0, 1] \times T^*N \rightarrow \mathbb{R}$  such that  $L_H$  intersects transversely both  $o_N$  and  $T_q^*N$ . We consider the Floer complexes

$$CF(L_H, o_N), \quad CF(o_N, T_q^*N), \quad CF(L_H, T_q^*N)$$

each of which carries filtration induced from the effective action function given in Proposition 7.3. We denote by  $\mathfrak{v}(\alpha)$  the level of the chain  $\alpha$  in any of these complexes.

More precisely,  $CF(L_H, o_N)$  is filtered by the effective functional

$$\mathcal{A}^{(1)}(\gamma) := \int \gamma^* \theta + h_H(\gamma(0)),$$

$CF^\mu(o_N, T_q^*N)$  by

$$\mathcal{A}^{(2)}(\gamma) := \int \gamma^* \theta,$$

and  $CF(L_H, T_q^*N)$  by

$$\mathcal{A}^{(0)}(\gamma) := \int \gamma^* \theta + h_H(\gamma(0))$$

respectively. We recall the readers that  $h_H$  is the potential of  $L_H$  and the zero function the potentials of  $o_N, T_q^*N$ .

We now consider the triangle product in the chain level, which we denote by

$$\mathfrak{m}_2 : CF(L_H, o_N) \otimes CF(o_N, T_q^*N) \rightarrow CF(L_H, T_q^*N) \quad (8.1)$$

following the general notation from [FOOO1], [Se]. This product is defined by considering all triples

$$x_1 \in L_H \cap o_N, \quad x_2 \in o_N \cap T_q^*N, \quad x_0 \in L_H \cap T_q^*N$$

with the polygonal Maslov index  $\mu(x_1, x_2; x_0)$  whose associated analytical index, or the virtual dimension of the moduli space

$$\mathcal{M}_3(D^2; x_1, x_2; x_0) := \widetilde{\mathcal{M}}_3(D^2; x_1, x_2; x_0) / PSL(2, \mathbb{R})$$

of  $J$ -holomorphic triangles, becomes zero and counting the number of elements thereof. The precise formula of the index is irrelevant to our discussion which, however, can be found in [Se], [FOOO2].

**Definition 8.2.** Let  $J = J(z)$  be a domain-dependent family of compatible almost complex structures with  $z \in D^2$ . We define the space  $\widetilde{\mathcal{M}}_3(D^2; x_1, x_2; x_0)$  by the pairs  $(w, (z_0, z_1, z_2))$  that satisfy the following:

- (1)  $w : D^2 \rightarrow T^*N$  is a continuous map satisfying  $\bar{\partial}_J w = 0$   $D^2 \setminus \{z_0, z_1, z_2\}$ ,
- (2) the marked points  $\{z_0, z_1, z_2\} \subset \partial D^2$  with counter-clockwise cyclic order,
- (3)  $w(z_1) = x_1$ ,  $w(z_2) = x_2$  and  $w(z_0) = x_0$ ,
- (4) the map  $w$  satisfies the Lagrangian boundary condition

$$w(\partial_1 D^2) \subset L_H, \quad w(\partial_2 D^2) \subset o_N, \quad w(\partial_3 D^2) \subset T_q^*N$$

where  $\partial_i D^2 \subset \partial D^2$  is the arc segment in between  $x_i$  and  $x_{i+1}$  ( $i \bmod 3$ ).

The general construction is by now well-known and e.g., given in [FOOO1]. In the current context of exact Lagrangian submanifolds, the detailed construction is also given in [Oh3] and [Se]. One important ingredient in relation to the study of the effect on the level of Floer chains under the product is the following (topological)

energy identity where the choice of the *effective* action functional plays a crucial role. For readers' convenience, we give its proof here.

**Proposition 8.2.** *Suppose  $w : D^2 \rightarrow T^*N$  be any smooth map with finite energy that satisfy all the conditions given in 8.2, but not necessarily  $J$ -holomorphic. We denote by  $c_x : [0, 1] \rightarrow T^*N$  the constant path with its value  $x \in T^*N$ . Then we have*

$$\int w^* \omega_0 = \mathcal{A}^{(1)}(c_{x_1}) + \mathcal{A}^{(2)}(c_{x_2}) - \mathcal{A}^{(0)}(c_{x_0}). \quad (8.2)$$

*Proof.* Recall  $\omega_0 = -d\theta$  and  $i^* \theta = dh_H$  on  $L_H$  and  $i^* \theta = 0$  on  $o_N$  and  $T_q^*N$  where  $i$ 's are the associated inclusion maps of  $L_H$ ,  $o_N$ ,  $T_q^*N \subset T^*N$  respectively. Therefore

$$\begin{aligned} \int_{D^2} w^* \omega_0 &= - \int_{\partial D^2} w^* \theta = - \int_{\partial_1 D^2} w^* \theta - \int_{\partial_2 D^2} w^* \theta - \int_{\partial D^2_3} w^* \theta \\ &= - \int_{\partial_1 D^2} w^* dh_H - 0 - 0 = h_H(w(z_1)) - h_H(w(z_2)) \\ &= h_H(x_1) - h_H(x_0) = \mathcal{A}^{(1)}(c_{x_1}) - \mathcal{A}^{(0)}(c_{x_0}) \\ &= \mathcal{A}^{(1)}(c_{x_1}) + \mathcal{A}^{(2)}(c_{x_2}) - \mathcal{A}^{(0)}(c_{x_0}). \end{aligned}$$

Here the last equality comes since  $\mathcal{A}^{(2)}(c_{x_2}) = \int c_{x_2}^* \theta = 0$ . This finishes the proof.  $\square$

An immediate corollary of this proposition from the definition of  $\mathbf{m}_2$  is that the map (8.1) restricts to

$$\mathbf{m}_2 : CF^\lambda(L_H, o_N) \otimes CF^\mu(o_N, T_q^*N) \rightarrow CF^{\lambda+\mu}(L_H, T_q^*N).$$

It is straightforward to check that this map satisfies

$$\partial(\mathbf{m}_2(x, y)) = \mathbf{m}_2(\partial(x), y) \pm \mathbf{m}_2(x, \partial(y))$$

and in turn induces the product map

$$*_F : HF^\lambda(L_H, o_N) \otimes HF^\mu(o_N, T_q^*N) \rightarrow HF^{\lambda+\mu}(L_H, T_q^*N) \quad (8.3)$$

in homology. This is because if  $w$  is  $J$ -holomorphic  $\int w^* \omega \geq 0$ . (We refer to [Oh3] and [FO] for the general construction of product map  $\mathbf{m}_2$  and to [Oh3], [MVZ] for the study of filtration. Similar study of filtration is also performed in [Sc], [Oh6] in the Hamiltonian Floer homology setting.)

With these preparations, we are ready to wrap-up the proof of Theorem 8.1:

*Proof of Theorem 8.1.* We consider a Floer cycle  $\alpha$  representing the fundamental class  $1^b = [M] \in HF(L_H, o_N)$  and  $\beta = \{q\}$  representing the unique generator of  $HF(o_N, T_q^*N) \cong \mathbb{Z}$ . Then by definition

$$\mathbf{v}(\alpha) \geq \rho^{lag}(H; 1), \quad \mathbf{v}(\beta) = \rho^{lag}(0; [q]) = 0.$$

Then its product cycle  $\mathbf{m}_2(\alpha, \beta) \in CF(L_H, T_q^*N)$  represents the homology class  $[q] \in CF(L_H, T_q^*N) \cong \mathbb{Z}$  and so  $\mathbf{v}(\mathbf{m}_2(\alpha, \beta)) \geq \rho^{lag}(H; \{q\}) = f_H(q)$  by definition of the latter. Applying the triangle inequality, we obtain

$$\mathbf{v}(\alpha) + 0 = \mathbf{v}(\alpha) + \mathbf{v}(\beta) \geq \mathbf{v}(\mathbf{m}_2(\alpha, \beta)) \geq \rho^{lag}(H; \{q\}) = f_H(q).$$

Therefore we have derived

$$\mathbf{v}(\alpha) \geq f_H(q)$$



for all cycle  $\alpha \in CF(L_H, o_N)$  representing  $[M]$ . By definition of  $\rho^{lag}(H; 1)$ , this proves

$$\rho^{lag}(H; 1) \geq f_H(q).$$

Since this holds for any point  $q \in N$ , we have proved  $\rho^{lag}(H; 1) \geq \max f_H$ .  $\square$

## 9. A HAMILTONIAN CONTINUITY THEOREM OF SPECTRAL CAPACITY

In this section, we prove the following Hamiltonian continuity of spectral capacity. The proof of this theorem is an adaptation to the Lagrangian context of the one used by Seyfaddini in his proof of Theorem 1 (or rather Corollary 1.2) [Sey]. The proof is also a variation of Ostrover's scheme used in [Os] and is an adaptation thereof. In our proof, we however use the Lagrangian analog to the notion of ' $\varepsilon$ -shiftability' introduced by Seyfaddini [Sey], instead of 'displaceability' used in [Os] and in other literature such as [EP], [U]. In the Lagrangian context here, the  $\varepsilon$ -shiftable domain is realized as the graph of  $df$  of a function  $f$  having no critical points on the corresponding domain. In this regard, it appears to the author that the notion of  $\varepsilon$ -shiftability becomes more geometric and intuitive in the Lagrangian context than in the Hamiltonian context.

Consider the subset

$$C_{crit}^\infty(N; B) = \{f \in C^\infty(N) \mid \text{Crit } f \subset \text{Int } B\}.$$

We recall the notation

$$\text{osc}_{C^0}(\phi_H^1; o_N) := \max \left\{ \max_{x \in o_N} d(\phi_H^1(x), x), \max_{x \in o_N} d(\phi_H^1)^{-1}(x), x) \right\}.$$

from (1.14).

**Theorem 9.1.** *Let  $\lambda_i = \phi_{H_i}$  where  $H_i \in \mathcal{PC}_{asc}^\infty$  is a sequence such that*

- (1) *there exists  $R > 0$  such that  $\text{supp } X_{H_i} \subset D^R(T^*N)$  for all  $i$  and  $s \in [0, 1]$ ,*
- (2) *There exists a closed ball  $B \subset N$  such that  $\text{supp } \phi_{H_i} \cap o_B = \emptyset$  for all  $i$  where we recall*

$$\text{supp } \phi_{H_i} = \bigcup_{t \in [0, 1]} \text{supp } \phi_{H_i}^t.$$

- (3) *There exists a uniform neighborhood  $T \supset o_B$  in  $T^*N$  such that  $\phi_{H_i}^1 \equiv \text{id}$  on  $T$  for all  $i$ 's.*

*Then if  $\lim_{i \rightarrow \infty} \text{osc}_{C^0}(\phi_{H_i}^1; o_N) = 0$ ,*

$$\lim_{i \rightarrow \infty} (\rho^{lag}(H_i; 1) - \rho^{lag}(L_{H_i}; [pt]^\#)) = 0.$$

The rest of the section is occupied by the proof of this theorem.

We fix a Riemannian metric  $g$  and the Levi-Civita connection on  $N$ . They naturally induces a metric on  $T^*N$ . Denote the latter metric on  $T^*N$  by  $\tilde{g}$  and the corresponding distance function by  $d(x, y)$  for  $x, y \in T^*N$ . We denote by  $D^r(T^*N)$  the disc bundle of  $T^*N$  of radius  $r$ .

The following is the well-known fact on this metric  $\tilde{g}$ , which can be easily checked.

**Lemma 9.2.** *The metric  $\tilde{g}$  carries following properties:*

- (1)  *$\tilde{g}$  is invariant under the reflection  $(q, p) \mapsto (q, -p)$  and in particular  $o_N$  is totally geodesic.*

- (2) *There exists a sufficiently small  $r = r(N, g) > 0$  depending only on  $(N, g)$  such that the following triangle inequality holds: Let  $x \in T^*N$  and denote  $x = (q(x), p(x))$ . Then*

$$d(o_{q(x)}, x) \geq \max\{|p(x)|, d(q, q(x))\} \geq |p(x)| \quad (9.1)$$

for all  $x \in D^r(T^*N)$  where  $|p(x)|$  is the norm on  $T_{q(x)}^*N$ .

We introduce a collection of the pairs  $(T, f)$  of a tubular neighborhood  $T \supset o_B$  in  $T^*N$  and a Morse function  $f \in C_{crit}^\infty(N; B, T)$  such that

- (1) all of its critical points contained in  $\text{Int } B$ ,
- (2)  $\text{Graph } df \subset D^r(T^*N)$  for  $r = r(N, g)$  given in Lemma 9.2,
- (3)  $\text{Graph}(df|_B) \subset T$ .

Denote by  $\mathcal{T}_B$  the set of all such pairs. We start with the following lemma

**Lemma 9.3.** *Let  $H \in \mathcal{PC}_{asc}^\infty$  in  $T^*N$  such that*

$$\text{supp } \phi_H \cap o_B = \emptyset, \quad (9.2)$$

and  $\phi_H^1 \equiv \text{id}$  on a neighborhood  $T \supset o_B$  in  $T^*N$ . Let  $(T, f) \in \mathcal{T}_B$  be given such that  $H$  satisfies  $\phi_H^1 \equiv \text{id}$ , and

$$\text{osc}_{C^0}(\phi_H^1; o_N) < C_1^-(f; N \setminus B, T)$$

where the constant  $C_1^-(f; N \setminus B, T)$  is defined below in (9.3). Then we have

$$L_f \cap o_N = \phi_H^1(L_f) \cap o_N$$

In particular all the Hamiltonian trajectories of  $H \# (f \circ \pi)$ , which have the form  $z_p^{H \# (f \circ \pi)}$  for some  $p \in L_f \cap o_N = \phi_H^1(L_f) \cap o_N$ , are constant equal to  $p$ .

*Proof.* In the proof, we will denote  $p \in N$  and the corresponding point in the zero section of  $T^*N$  by  $o_p$  for the notational consistency.

By the choice of the pair  $(T, f) \in \mathcal{T}_B$ , we have

$$\min \left\{ \min_{p \in N \setminus B} |df(p)|, d_H(N \setminus B, \text{Crit } f) \right\} > 0.$$

where  $d_H(N \setminus B, \text{Crit } f)$  is the Hausdorff distance. We define a positive constant

$$C_1^-(f; N \setminus B) := \min \left\{ \min_{p \in N \setminus B} |df(p)|, d_H(N \setminus B, \text{Crit } f) \right\} \quad (9.3)$$

By definition of  $C_1^-(f; N \setminus B, T)$ , if  $q \in N \setminus B$ , we have

$$|df(q)|, d(q, \text{Crit } f) \geq C_1^-(f; N \setminus B, T) > 0. \quad (9.4)$$

Obviously we have  $\text{Crit } f = L_f \cap o_B \subset \phi_H^1(L_f) \cap o_N$  since we assume  $\phi_H^1 \equiv \text{id}$  on a neighborhood,  $T$ , of  $o_B \supset \text{Crit } f$ .

We will now prove the opposite inclusion  $\phi_H^1(L_f) \cap o_N \subset L_f \cap o_B$ . Suppose  $o_p \in \phi_H^1(L_f) \cap o_N$ . Then we have  $(\phi_H^1)^{-1}(o_p) \in L_f$ .

Consider first the case  $p \in B$ . In this case since we assume  $\phi_H^1 = \text{id}$  on a neighborhood of  $o_B$ , it in particular implies  $o_p = (\phi_H^1)^{-1}(o_p)$  for all  $i$  and hence  $o_p \in o_B \cap L_f \cong \text{Crit } f$ .

Now we will show that  $p$  cannot lie in  $N \setminus B$ . Suppose  $p \in N \setminus B$  to the contrary and write

$$(\phi_H^1)^{-1}(o_p) = df(p')$$

for some  $p' \in N$ . Therefore

$$d(o_p, df(p')) = d(o_p, (\phi_H^1)^{-1}(o_p)) \leq \text{osc}_{C^0}(\phi_H^1; o_N).$$

Furthermore we also have  $|df(p')| \leq d(o_p, df(p'))$  by Lemma 9.2 since  $\text{Graph } df \subset D^*(T^*N)$ . Therefore we have shown

$$|df(p')| \leq \text{osc}_{C^0}(\phi_H^1; o_N) < C_1^-(f; N \setminus B, T). \quad (9.5)$$

This in particular implies  $(\phi_H^1)^{-1}(o_p) = df(p')$  must lie in  $\text{Graph } df|_B \subset T$  for otherwise  $|df(p')| \geq C_1^-(f; N \setminus B, T)$  by definition of  $C_1^-(f; N \setminus B, T)$  which would contradict to (9.5).

This in turn implies  $(\phi_H^1)^{-1}(o_p) \in T$ . But  $\phi_H^1$  is assumed to be the identity map on  $T$  and hence follows

$$o_p = (\phi_H^1)^{-1}(o_p) = df(p').$$

In particular  $df(p') \in o_N$  and so  $p' \in \text{Crit } f$  and hence  $o_{p'} = df(p')$ . This implies  $p = p'$  and so  $d(p, \text{Crit } f) = 0$ , i.e.,  $p \in \text{Crit } f \subset B$ , a contradiction to the hypothesis  $p \in N \setminus B$ .

Therefore  $p$  cannot lie in  $N \setminus B$  and hence proves  $o_p \in o_B \cap L_f \cong \text{Crit } f$  for any  $o_p \in \phi_H^1(L_f) \cap o_N$ . This then finishes the proof of the first statement

$$L_f \cap o_N = \phi_H^1(L_f) \cap o_N. \quad (9.6)$$

To prove the second statement, we recall the definition

$$z_p^{H \# f \circ \pi}(t) = \phi_{H \# f \circ \pi}^t((\phi_{H \# f \circ \pi}^1)^{-1}(p))$$

and so  $z_p^{H \# f \circ \pi}(1) = p$ . But we have  $df(p) = 0$  and  $(\phi_H^1)^{-1}(o_p) = o_p$  since

$$p \in \phi_H^1(L_f) \cap o_N = L_f \cap o_N \subset o_B \cap \text{Crit } f$$

and  $\phi_H^1 \equiv id$  near  $p$ . Therefore

$$(\phi_{H \# f \circ \pi}^1)^{-1}(o_p) = (\phi_{f \circ \pi}^1)^{-1}(\phi_H^1)^{-1}(o_p) = o_p.$$

On the other hand  $\phi_H^t \equiv id$  on a neighborhood  $T'_i \supset o_B$  in  $T^*N$  since we assume  $\text{supp } \phi_H \cap o_B = \emptyset$ . Therefore

$$\begin{aligned} z_p^{H \# f \circ \pi}(t) &= \phi_{H \# f \circ \pi}^t((\phi_{H \# f \circ \pi}^1)^{-1}(o_p)) = \phi_{H \# f \circ \pi}^t(o_p) \\ &= \phi_H^t(\phi_{f \circ \pi}^t(o_p)) = \phi_H^t(o_p) = o_p \end{aligned}$$

since  $df(p) = 0$  and  $\phi_H^t(o_p) = o_p$  for all  $t \in [0, 1]$ . This finishes the proof.  $\square$

**Remark 9.1.** We would like to mention that in the above proof, the choice of the neighborhood  $T'_i$  may depend on  $i$ 's and so may not be able to choose a uniform neighborhood  $T'$  independent of  $i$ 's.

Motivated by the proof of this proposition, we introduce a collection, denoted by  $C_{crit}^\infty(N; B, T) \subset C^\infty(N)$ , of Morse functions  $f$  satisfying the condition in this lemma. We define the subset  $C_{crit}^\infty(N; B) \subset C^\infty(N)$  to be the union

$$C_{crit}^\infty(N; B) = \bigcup_T C_{crit}^\infty(N; B, T).$$

It is easy to check that  $C_{crit}^\infty(N; B, T) \neq \emptyset$  for any such  $T \supset o_B$  by considering the  $\lambda f$  for a sufficiently small  $\lambda > 0$  for any given Morse function  $f$  with  $\text{Crit } f \subset \text{Int } B$ .

**Lemma 9.4.** *For any  $f \in C_{crit}^\infty(N; B, T)$ , the constant  $C_1^-(f; N \setminus B, T)$  satisfies*

$$C_1^-(\lambda f; N \setminus B, T) = \min_{p \in N \setminus B} |d(\lambda f)(p)| \quad (9.7)$$

whenever  $\lambda$  is so small that

$$\min_{p \in N \setminus B} |d(\lambda f)(p)| < d_H(N \setminus T, \text{Crit } f).$$

In particular, we have

$$\lambda C_1^-(f; N \setminus B, T) = C_1^-(\lambda f; N \setminus B, T)$$

for such  $\lambda$ 's.

*Proof.* First note that

$$\min_{p \in N \setminus B} |\lambda df(p)| = \lambda \min_{p \in N \setminus B} |df(p)| \rightarrow 0$$

as  $\lambda \rightarrow 0$  but  $d_H(N \setminus B, \text{Crit}(\lambda f))$  is independent of  $\lambda$ . Therefore the minimum in the definition

$$C_1^-(\lambda f; N \setminus B, T) = \min \left\{ \min_{p \in N \setminus B} |d(\lambda f)(p)|, d_H(N \setminus B, \text{Crit}(\lambda f)) \right\}$$

is realized by  $\min_{p \in N \setminus B} |d(\lambda f)(p)|$  for all sufficiently small  $\lambda$ . Then the lemma follows.  $\square$

The following proposition is a crucial ingredient of the proof, which is a variation of Proposition 2.6 [Os], Proposition 3.3 [EP], Proposition 3.1 [U] and Proposition 2.3 [Sey].

**Proposition 9.5.** *Let  $H \in \mathcal{PC}_{asc}^\infty$  in  $T^*N$  such that*

$$\text{supp } \phi_H \cap o_B = \emptyset. \quad (9.8)$$

*Take any  $f \in C_{crit}^\infty(N; B)$  such that*

$$\text{osc}_{C^0}(\phi_H^1; o_N) < C_1^-(f; N \setminus B, T). \quad (9.9)$$

*Then*

$$\rho^{lag}(H; 1) - \rho^{lag}(H; [pt]^\#) \leq 2 \text{osc } f. \quad (9.10)$$

*Proof.* Denote  $L_f := \text{Graph } df$ ,  $L_t = \phi_H^t(L_f) = \phi_H^t(\text{Graph } df)$ . Note that the condition (9.8) implies

$$H_t|_B \equiv c_B(t) \quad (9.11)$$

for a function  $c_B = c_B(t)$  depending only on  $t$  but not on  $x \in B$ .

The following lemma is the analogue of Lemma 5.1 [Os].

**Lemma 9.6.**

$$\rho^{lag}(H \# f; 1) - \rho^{lag}(H \# f; [pt]^\#) \leq \text{osc } f. \quad (9.12)$$

*Proof.* By the spectrality of  $\rho^{lag}(\cdot, 1)$  in general, we have

$$\begin{aligned} \rho^{lag}(H \# f \circ \pi; 1) &= \mathcal{A}_{(H \# f \circ \pi)}^{cl} \left( z_{p_-}^{H \# f \circ \pi} \right), \\ \rho^{lag}(H \# f \circ \pi; [pt]^\#) &= \mathcal{A}_{(H \# f \circ \pi)}^{cl} \left( z_{p_+}^{H \# f \circ \pi} \right) \end{aligned}$$

for some  $p_{\pm} \in L_f \cap o_N$ . Using the second statement of Lemma 9.3, we compute

$$\begin{aligned}
& \mathcal{A}_{(H \# f \circ \pi)}^{cl} \left( z_{p_+}^{H \# f \circ \pi} \right) - \mathcal{A}_{(H \# f \circ \pi)}^{cl} \left( z_{p_-}^{H \# f \circ \pi} \right) \\
&= - \int_0^1 (H \# f \circ \pi)(t, p_+) dt + \int_0^1 (H \# f \circ \pi)(t, p_-) dt \\
&= - \int_0^1 c_B(t) dt - f(p_+) + \int_0^1 c_B(t) dt + f(p_-) \\
&= -f(p_+) + f(p_-) \leq \max f - \min f = \text{osc } f.
\end{aligned}$$

Here for the equality in the line next to the last, we use the identity

$$(H \# f \circ \pi)(t, p_{\pm}) = H(t, p_{\pm}) + f(\phi_H^t(p_{\pm})) = c_B(t) + f(p_{\pm}).$$

This finishes the proof.  $\square$

On the other hand, we have

$$\phi_H^1(L_f) = \phi_H^1(\phi_{f \circ \pi}^1(o_N)) = \phi_{H \# f \circ \pi}^1(o_N)$$

and so by the triangle inequality, Proposition 7.5,

$$\begin{aligned}
\rho^{lag}(H \# (f \circ \pi); 1) &\geq \rho^{lag}(H; 1) - \rho^{lag}(-f \circ \pi; 1) \\
\rho^{lag}(H \# (f \circ \pi); [pt]^{\#}) &\leq \rho^{lag}(H; [pt]^{\#}) + \rho^{lag}(f \circ \pi; 1).
\end{aligned}$$

(One can also use Proposition 7.6 using the concatenation  $H * (f \circ \pi)$  instead. Here  $f \circ \pi$  is not boundary flat, which is required in Proposition 7.6, but one can always reparameterize the flow  $t \mapsto \phi_{f \circ \pi}^t$  by multiplying  $\chi'(t)$  to  $f \circ \pi$  so that the perturbation is as small as we want in  $L^{(1, \infty)}$ -topology which in turn perturbs  $\rho$  slightly. See Lemma 5.2 [Oh4], Remark 2.5 [MVZ] for the precise statement on this approximation procedure, or Appendix of the present paper. This enables us to apply the triangle inequality in Proposition 7.6 in the current context.)

Therefore subtracting the second inequality from the first and using the identity

$$\rho^{lag}(-f \circ \pi; 1) = \max f, \quad \rho^{lag}(f \circ \pi; 1) = -\min f$$

(see [?], [Oh3] for its proof), we obtain

$$\begin{aligned}
& \rho^{lag}(H \# (f \circ \pi); 1) - \rho^{lag}(H \# (f \circ \pi); [pt]^{\#}) \\
&\geq \rho^{lag}(H; 1) - \rho^{lag}(H; [pt]^{\#}) - (\max f - \min f)
\end{aligned}$$

which in turn gives rise to

$$\begin{aligned}
\rho^{lag}(H; 1) - \rho^{lag}(H; [pt]^{\#}) &\leq \rho^{lag}(H \# (f \circ \pi); 1) - \rho^{lag}(H \# (f \circ \pi); [pt]^{\#}) \\
&\quad + (\max f - \min f) \\
&\leq 2 \text{osc } f.
\end{aligned}$$

We have finished the proof of the proposition.  $\square$

We now go back to the proof of Theorem 9.1.

Consider the elements  $H_i$  in the given sequence that satisfy (9.8).  $\phi_{H_i}^1 \equiv id$  on a uniform  $T \supset o_B$ , and the oscillation  $\text{osc}_{C^0}(\phi_{H_i}^1; o_N)$  can be made arbitrarily small by letting  $i \rightarrow \infty$ .

If  $\text{osc}_{C^0}(\phi_{H_i}^1; o_N) = 0$  for all sufficiently large  $i$ 's, we have  $\phi_{H_i}^1(o_N) = o_N$  and so  $\rho^{lag}(H_i; 1) - \rho^{lag}(H_i; [pt]^\#) = 0$  for which (9.10) obviously holds. Therefore we assume that there exists a subsequence, again denoted by  $H_i$ , such that  $\text{osc}_{C^0}(\phi_{H_i}^1; o_N) \neq 0$ .

Since  $\text{supp } \phi_{H_i} \cap o_B = \emptyset$  and  $\phi_{H_i}^1 \equiv 0$  on  $T$  for all  $i$ , and  $\text{osc}_{C^0}(\phi_{H_i}^1; o_N) \rightarrow 0$  as  $i \rightarrow \infty$ , we have

$$\text{osc}_{C^0}(\phi_{H_i}^1; o_N) < C_1^-(f; N \setminus B, T)$$

eventually. Recall from Lemma 9.3 that the choice of  $f$  depends only on the ball  $B$  and the neighborhood  $T \subset B$  in  $T^*N$ . Then we choose  $\lambda_i > 0$  such that

$$\text{osc}_{C^0}(\phi_{H_i}^1; o_N) = \lambda_i C_1^-(f; N \setminus B, T)$$

i.e.,

$$\lambda_i = \frac{\text{osc}_{C^0}(\phi_{H_i}^1; o_N)}{C_1^-(f; N \setminus B, T)}.$$

Since  $\text{osc}_{C^0}(\phi_{H_i}^1; o_N) \rightarrow 0$ ,  $\lambda_i \rightarrow 0$ . Obviously we have

$$\text{osc}_{C^0}(\phi_{H_i}^1; o_N) < (\lambda_i + \varepsilon) C_1^-(f; N \setminus B, T)$$

for all  $\varepsilon > 0$ . Consider sufficiently large  $i$ 's so that

$$\min_{p \in N \setminus B} |d(\lambda_i f)(p)| < d_H(N \setminus B, \text{Crit } f)$$

and hence

$$\lambda_i C_1^-(f; N \setminus B, T) = C_1^-(\lambda_i f; N \setminus B, T)$$

by Lemma 9.4.

Now we fix any such  $i$ . Lemma 9.4 also implies

$$(\lambda_i + \varepsilon) C_1^-(f; N \setminus B, T) = C_1^-((\lambda_i + \varepsilon)f; N \setminus B, T)$$

for all small  $\varepsilon > 0$  such that

$$\min_{p \in N \setminus B} |(\lambda_i + \varepsilon) df(p)| < d(N \setminus B, \text{Crit } f).$$

For example, we can choose any  $\varepsilon > 0$  so that

$$0 < \varepsilon < \frac{d(N \setminus B, \text{Crit } f)}{\min_{p \in N \setminus B} |df(p)|}. \quad (9.13)$$

Note that the upper bound does not depend on  $i$ 's at all.

Since (9.10) holds for any pair  $H, f$  that satisfy (9.8) and (9.9), applying it to the pair  $(H_i, (\lambda_i + \varepsilon)f)$  for  $T \supset B$  chosen above independently of  $i$ 's, we derive

$$\begin{aligned} \rho^{lag}(H_i; 1) - \rho^{lag}(H_i; [pt]^\#) &\leq 2 \text{osc}((\lambda_i + \varepsilon)f) = 2(\lambda_i + \varepsilon) \text{osc } f \\ &= 2 \left( \frac{\text{osc}_{C^0}(\phi_{H_i}^1; o_N)}{C_1^-(f; N \setminus B, T)} + \varepsilon \right) \text{osc } f. \end{aligned}$$

Since this holds for all  $\varepsilon > 0$  satisfying (9.13), it follows

$$0 \leq \rho^{lag}(H_i; 1) - \rho^{lag}(H_i; [pt]^\#) \leq 2 \left( \frac{\text{osc } f}{C_1^-(f; N \setminus B, T)} \right) \text{osc}_{C^0}(\phi_{H_i}^1; o_N) \quad (9.14)$$

letting  $\varepsilon \rightarrow 0$ .

This inequality in particular finishes the proof of Theorem 9.1.  $\square$

The following upper bound of the spectral capacity involving the  $C^0$ -metric  $\text{osc}_{C^0}(\phi_H^1; o_N)$  has been obtained in the course of the above proof, which has some independent interest in its own right.

**Theorem 9.7.** *Let  $B \subset N$  be a closed ball and  $(T, f) \in \mathcal{T}_B$ . Consider the set of Hamiltonians  $H \in \mathcal{P}C_{asc,0}^\infty$  satisfying  $\text{supp } \phi_H \cap o_B = \emptyset$  and assume*

$$\text{osc}_{C^0}(\phi_H^1; o_N) < C_1^-(f; N \setminus B, T).$$

*Then we have*

$$\frac{\rho^{\text{lag}}(H; 1) - \rho^{\text{lag}}(H; [pt]^\#)}{\text{osc}_{C^0}(\phi_H^1; o_N)} \leq \frac{2 \text{osc} f}{C_1^-(f; N \setminus B, T)}. \quad (9.15)$$

The following question seems to be an interesting question to ask in regard to the precise estimate of the upper bound in this theorem.

**Question 9.2.** For given  $H$  satisfying the condition in Theorem 9.7, what is an optimal estimate of the constant  $\frac{2 \text{osc} f}{C_1^-(f; N \setminus B, T)}$  in terms of  $B$ ,  $T$  and  $H$ ? For example, can we obtain an upper bound independent of  $B$  or  $T$ ?

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